8 Conformal Mappings

> The results I found for polygons can be extended under very general assumptions. I have undertaken this research because it is a step towards a deeper understanding of the mapping problem, for which not much has happened since Riemann's inaugural dissertation; this, even though the theory of mappings, with its close connection with the fundamental theorems of Riemann's function theory, deserves in the highest degree to be developed further.
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> *E. B. Christoffel,* 1870

The problems and ideas we present in this chapter are more geometric in nature than the ones we have seen so far. In fact, here we will be primarily interested in mapping properties of holomorphic functions. In particular, most of our results will be "global," as opposed to the more "local" analytical results proved in the first three chapters. The motivation behind much of our presentation lies in the following simple question:

> Given two open sets U and V in $\mathbb{C}$, does there exist a holomorphic bijection between them?

By a holomorphic bijection we simply mean a function that is both holomorphic and bijective. (It will turn out that the inverse map is then automatically holomorphic.) A solution to this problem would permit a transfer of questions about analytic functions from one open set with little geometric structure to another with possibly more useful properties. The prime example consists in taking $V = \mathbb{D}$ the unit disc, where many ideas have been developed to study analytic functions.[1] In fact, since the disc seems to be the most fruitful choice for V we are led to a variant of the above question:

> Given an open subset Ω of $\mathbb{C}$, what conditions on Ω guarantee that there exists a holomorphic bijection from Ω to $\mathbb{D}$?

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[1] For the corresponding problem when $V = \mathbb{C}$, the solution is trivial: only $U = \mathbb{C}$ is possible. See Exercise 14 in Chapter 3.

In some instances when a bijection exists it can be given by explicit formulas, and we turn to this aspect of the theory first. For example, the upper half-plane can be mapped by a holomorphic bijection to the disc, and this is given by a fractional linear transformation. From there, one can construct many other examples, by composing simple maps already encountered earlier, such as rational functions, trigonometric functions, logarithms, etc. As an application, we discuss the consequence of these constructions to the solution of the Dirichlet problem for the Laplacian in some particular domains.

Next, we pass from the specific examples to prove the first general result of the chapter, namely the Schwarz lemma, with an immediate application to the determination of all holomorphic bijections ("automorphisms" of the disc to itself). These are again given by fractional linear transformations.

Then comes the heart of the matter: the Riemann mapping theorem, which states that Ω can be mapped to the unit disc whenever it is simply connected and not all of $\mathbb{C}$. This is a remarkable theorem, since little is assumed about Ω, not even regularity of its boundary $\partial\Omega$. (After all, the boundary of the disc is smooth.) In particular, the interiors of triangles, squares, and in fact any polygon can be mapped via a bijective holomorphic function to the disc. A precise description of the mapping in the case of polygons, called the Schwarz-Christoffel formula, will be taken up in the last section of the chapter. It is interesting to note that the mapping functions for rectangles are given by "elliptic integrals," and these lead to doubly-periodic functions. The latter are the subject of the next chapter.

1 Conformal equivalence and examples

We fix some terminology that we shall use in the rest of this chapter. A bijective holomorphic function $f : U \to V$ is called a **conformal map** or **biholomorphism**. Given such a mapping f, we say that U and V are **conformally equivalent** or simply **biholomorphic**. An important fact is that the inverse of f is then automatically holomorphic.

Proposition 1.1 *If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.*

Proof. We argue by contradiction, and suppose that $f'(z_0) = 0$ for some $z_0 \in U$. Then

$$f(z) - f(z_0) = a(z - z_0)^k + G(z) \quad \text{for all } z \text{ near } z_0,$$

with $a \neq 0$, $k \geq 2$ and G vanishing to order $k+1$ at z_0. For sufficiently small w, we write

$$f(z) - f(z_0) - w = F(z) + G(z), \quad \text{where } F(z) = a(z - z_0)^k - w.$$

Since $|G(z)| < |F(z)|$ on a small circle centered at z_0, and F has at least two zeros inside that circle, Rouché's theorem implies that $f(z) - f(z_0) - w$ has at least two zeros there. Since $f'(z) \neq 0$ for all $z \neq z_0$ but sufficiently close to z_0 it follows that the roots of $f(z) - f(z_0) - w$ are distinct, hence f is not injective, a contradiction.

Now let $g = f^{-1}$ denote the inverse of f on its range, which we can assume is V. Suppose $w_0 \in V$ and w is close to w_0. Write $w = f(z)$ and $w_0 = f(z_0)$. If $w \neq w_0$, we have

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{w - w_0}{g(w) - g(w_0)}} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}.$$

Since $f'(z_0) \neq 0$, we may let $z \to z_0$ and conclude that g is holomorphic at w_0 with $g'(w_0) = 1/f'(g(w_0))$.

From this proposition we conclude that two open sets U and V are conformally equivalent if and only if there exist holomorphic functions $f : U \to V$ and $g : V \to U$ such that $g(f(z)) = z$ and $f(g(w)) = w$ for all $z \in U$ and $w \in V$.

We point out that the terminology adopted here is not universal. Some authors call a holomorphic map $f : U \to V$ conformal if $f'(z) \neq 0$ for all $z \in U$. This definition is clearly less restrictive than ours; for example, $f(z) = z^2$ on the punctured disc $\mathbb{C} - \{0\}$ satisfies $f'(z) \neq 0$, but is not injective. However, the condition $f'(z) \neq 0$ is tantamount to f being a local bijection (Exercise 1). There is a geometric consequence of the condition $f'(z) \neq 0$ and it is at the root of this discrepency of terminology in the definitions. A holomorphic map that satisfies this condition preserves angles. Loosely speaking, if two curves γ and η intersect at z_0, and α is the oriented angle between the tangent vectors to these curves, then the image curves $f \circ \gamma$ and $f \circ \eta$ intersect at $f(z_0)$, and their tangent vectors form the same angle α. Problem 2 develops this idea.

We begin our study of conformal mappings by looking at a number of specific examples. The first gives the conformal equivalence between the unit disc and the upper half-plane, which plays an important role in many problems.

1.1 The disc and upper half-plane

The upper half-plane, which we denote by $\mathbb{H}$, consists of those complex numbers with positive imaginary part; that is,

$$\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$

A remarkable fact, which at first seems surprising, is that the unbounded set $\mathbb{H}$ is conformally equivalent to the unit disc. Moreover, an explicit formula giving this equivalence exists. Indeed, let

$$F(z) = \frac{i - z}{i + z} \quad \text{and} \quad G(w) = i\frac{1 - w}{1 + w}.$$

Theorem 1.2 *The map $F : \mathbb{H} \to \mathbb{D}$ is a conformal map with inverse $G : \mathbb{D} \to \mathbb{H}$.*

Proof. First we observe that both maps are holomorphic in their respective domains. Then we note that any point in the upper half-plane is closer to i than to $-i$, so $|F(z)| < 1$ and F maps $\mathbb{H}$ into $\mathbb{D}$. To prove that G maps into the upper half-plane, we must compute $\text{Im}(G(w))$ for $w \in \mathbb{D}$. To this end we let $w = u + iv$, and note that

$$\begin{aligned}
\text{Im}(G(w)) &= \text{Re}\left(\frac{1 - u - iv}{1 + u + iv}\right) \\
&= \text{Re}\left(\frac{(1 - u - iv)(1 + u - iv)}{(1 + u)^2 + v^2}\right) \\
&= \frac{1 - u^2 - v^2}{(1 + u)^2 + v^2} > 0
\end{aligned}$$

since $|w| < 1$. Therefore G maps the unit disc to the upper half-plane. Finally,

$$F(G(w)) = \frac{i - i\frac{1-w}{1+w}}{i + i\frac{1-w}{1+w}} = \frac{1 + w - 1 + w}{1 + w + 1 - w} = w,$$

and similarly $G(F(z)) = z$. This proves the theorem.

An interesting aspect of these functions is their behavior on the boundaries of our open sets.[2] Observe that F is holomorphic everywhere on $\mathbb{C}$

[2]The boundary behavior of conformal maps is a recurrent theme that plays an important role in this chapter.

except at $z = -i$, and in particular it is continuous everywhere on the boundary of $\mathbb{H}$, namely the real line. If we take $z = x$ real, then the distance from x to i is the same as the distance from x to $-i$, therefore $|F(x)| = 1$. Thus F maps $\mathbb{R}$ onto the boundary of $\mathbb{D}$. We get more information by writing

$$F(x) = \frac{i-x}{i+x} = \frac{1-x^2}{1+x^2} + i\frac{2x}{1+x^2},$$

and parametrizing the real line by $x = \tan t$ with $t \in (-\pi/2, \pi/2)$. Since

$$\sin 2a = \frac{2\tan a}{1+\tan^2 a} \quad \text{and} \quad \cos 2a = \frac{1-\tan^2 a}{1+\tan^2 a},$$

we have $F(x) = \cos 2t + i \sin 2t = e^{i2t}$. Hence the image of the real line is the arc consisting of the circle omitting the point -1. Moreover, as x travels from $-\infty$ to ∞, $F(x)$ travels along that arc starting from -1 and first going through that part of the circle that lies in the lower half-plane.

The point -1 on the circle corresponds to the "point at infinity" of the upper half-plane.

Remark. Mappings of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where a, b, c, and d are complex numbers, and where the denominator is assumed not to be a multiple of the numerator, are usually referred to as **fractional linear transformations**. Other instances occur as the automorphisms of the disc and of the upper half-plane in Theorems 2.1 and 2.4.

1.2 Further examples

We gather here several illustrations of conformal mappings. In certain cases we discuss the behavior of the map on the boundary of the relevant domain. Some of the mappings are pictured in Figure 1.

Example 1. Translations and dilations provide the first simple examples. Indeed, if $h \in \mathbb{C}$, the translation $z \mapsto z + h$ is a conformal map from $\mathbb{C}$ to itself whose inverse is $w \mapsto w - h$. If h is real, then this translation is also a conformal map from the upper half-plane to itself.

For any non-zero complex number c, the map $f : z \mapsto cz$ is a conformal map from the complex plane to itself, whose inverse is simply $g : w \mapsto c^{-1}w$. If c has modulus 1, so that $c = e^{i\varphi}$ for some real φ, then f is a **rotation** by φ. If $c > 0$ then f corresponds to a dilation. Finally, if $c < 0$ the map f consists of a dilation by $|c|$ followed by a rotation of π.

EXAMPLE 2. If n is a positive integer, then the map $z \mapsto z^n$ is conformal from the sector $S = \{z \in \mathbb{C} : 0 < \arg(z) < \pi/n\}$ to the upper half-plane. The inverse of this map is simply $w \mapsto w^{1/n}$, defined in terms of the principal branch of the logarithm.

More generally, if $0 < \alpha < 2$ the map $f(z) = z^\alpha$ takes the upper half-plane to the sector $S = \{w \in \mathbb{C} : \ 0 < \arg(w) < \alpha\pi\}$. Indeed, if we choose the branch of the logarithm obtained by deleting the positive real axis, and $z = re^{i\theta}$ with $r > 0$ and $0 < \theta < \pi$, then

$$f(z) = z^\alpha = |z|^\alpha e^{i\alpha\theta}.$$

Therefore f maps $\mathbb{H}$ into S. Moreover, a simple verification shows that the inverse of f is given by $g(w) = w^{1/\alpha}$, where the branch of the logarithm is chosen so that $0 < \arg w < \alpha\pi$.

By composing the map just discussed with the translations and rotations in the previous example, we may map the upper half-plane conformally to any (infinite) sector in $\mathbb{C}$.

Let us note the boundary behavior of f. If x travels from $-\infty$ to 0 on the real line, then $f(x)$ travels from $\infty e^{i\alpha\pi}$ to 0 on the half-line determined by $\arg z = \alpha\pi$. As x goes from 0 to ∞ on the real line, the image $f(x)$ goes from 0 to ∞ on the real line as well.

EXAMPLE 3. The map $f(z) = (1 + z)/(1 - z)$ takes the upper half-disc $\{z = x + iy : \ |z| < 1 \text{ and } y > 0\}$ conformally to the first quadrant $\{w = u + iv : \ u > 0 \text{ and } v > 0\}$. Indeed, if $z = x + iy$ we have

$$f(z) = \frac{1 - (x^2 + y^2)}{(1 - x)^2 + y^2} + i\frac{2y}{(1 - x)^2 + y^2},$$

so f maps the half-disc in the upper half-plane into the first quadrant. The inverse map, given by $g(w) = (w - 1)/(w + 1)$, is clearly holomorphic in the first quadrant. Moreover, $|w + 1| > |w - 1|$ for all w in the first quadrant because the distance from w to -1 is greater than the distance from w to 1; thus g maps into the unit disc. Finally, an easy calculation shows that the imaginary part of $g(w)$ is positive whenever w is in the first quadrant. So g transforms the first quadrant into the desired half-disc and we conclude that f is conformal because g is the inverse of f.

To examine the action of f on the boundary, note that if $z = e^{i\theta}$ belongs to the upper half-circle, then

$$f(z) = \frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{e^{-i\theta/2}+e^{i\theta/2}}{e^{-i\theta/2}-e^{i\theta/2}} = \frac{i}{\tan(\theta/2)}.$$

As θ travels from 0 to π we see that $f(e^{i\theta})$ travels along the imaginary axis from infinity to 0. Moreover, if $z = x$ is real, then

$$f(z) = \frac{1+x}{1-x}$$

is also real; and one sees from this, that f is actually a bijection from $(-1,1)$ to the positive real axis, with $f(x)$ increasing from 0 to infinity as x travels from -1 to 1. Note also that $f(0) = 1$.

Example 4. The map $z \mapsto \log z$, defined as the branch of the logarithm obtained by deleting the negative imaginary axis, takes the upper half-plane to the strip $\{w = u + iv : u \in \mathbb{R},\ 0 < v < \pi\}$. This is immediate from the fact that if $z = re^{i\theta}$ with $-\pi/2 < \theta < 3\pi/2$, then by definition,

$$\log z = \log r + i\theta.$$

The inverse map is then $w \mapsto e^w$.

As x travels from $-\infty$ to 0, the point $f(x)$ travels from $\infty + i\pi$ to $-\infty + i\pi$ on the line $\{x + i\pi : -\infty < x < \infty\}$. When x travels from 0 to ∞ on the real line, its image $f(x)$ then goes from $-\infty$ to ∞ along the reals.

Example 5. With the previous example in mind, we see that $z \mapsto \log z$ also defines a conformal map from the half-disc $\{z = x + iy : |z| < 1,\ y > 0\}$ to the half-strip $\{w = u + iv : u < 0,\ 0 < v < \pi\}$. As x travels from 0 to 1 on the real line, then $\log x$ goes from $-\infty$ to 0. When x goes from 1 to -1 on the half-circle in the upper half-plane, then the point $\log x$ travels from 0 to πi on the vertical segment of the strip. Finally, as x goes from -1 to 0, the point $\log x$ goes from πi to $-\infty + i\pi$ on the top half-line of the strip.

Example 6. The map $f(z) = e^{iz}$ takes the half-strip $\{z = x + iy : -\pi/2 < x < \pi/2,\ \ y > 0\}$ conformally to the half-disc $\{w = u + iv : |w| < 1,\ u > 0\}$. This is immediate from the fact that if $z = x + iy$, then

$$e^{iz} = e^{-y}e^{ix}.$$

If x goes from $\pi/2 + i\infty$ to $\pi/2$, then $f(x)$ goes from 0 to i, and as x goes from $\pi/2$ to $-\pi/2$, then $f(x)$ travels from i to $-i$ on the half-circle. Finally, as x goes from $-\pi/2$ to $-\pi/2 + i\infty$, we see that $f(x)$ travels from $-i$ back to 0.

The mapping f is closely related to the inverse of the map in Example 5.

EXAMPLE 7. The function $f(z) = -\frac{1}{2}(z + 1/z)$ is a conformal map from the half-disc $\{z = x + iy : |z| < 1,\ y > 0\}$ to the upper half-plane (Exercise 5).

The boundary behavior of f is as follows. If x travels from 0 to 1, then $f(x)$ goes from ∞ to 1 on the real axis. If $z = e^{i\theta}$, then $f(z) = \cos\theta$ and as x travels from 1 to -1 along the unit half-circle in the upper half-plane, the $f(x)$ goes from 1 to -1 on the real segment. Finally, when x goes from -1 to 0, $f(x)$ goes from -1 to $-\infty$ along the real axis.

EXAMPLE 8. The map $f(z) = \sin z$ takes the upper half-plane conformally onto the half-strip $\{w = x + iy : -\pi/2 < x < \pi/2\ y > 0\}$. To see this, note that if $\zeta = e^{iz}$, then

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{-1}{2}\left(i\zeta + \frac{1}{i\zeta}\right),$$

and therefore f is obtained first by applying the map in Example 6, then multiplying by i (that is, rotating by $\pi/2$), and finally applying the map in Example 7.

As x travels from $-\pi/2 + i\infty$ to $-\pi/2$, the point $f(x)$ goes from $-\infty$ to -1. When x is real, between $-\pi/2$ and $\pi/2$, then $f(x)$ is also real between -1 and 1. Finally, if x goes from $\pi/2$ to $\pi/2 + i\infty$, then $f(x)$ travels from 1 to ∞ on the real axis.

1.3 The Dirichlet problem in a strip

The Dirichlet problem in the open set Ω consists of solving

$$\text{(1)} \qquad \begin{cases} \triangle u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $\triangle$ denotes the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and f is a given function on the boundary of Ω. In other words, we wish to find a harmonic function in Ω with prescribed boundary values f. This problem was already considered in Book I in the cases where Ω is the unit disc or the

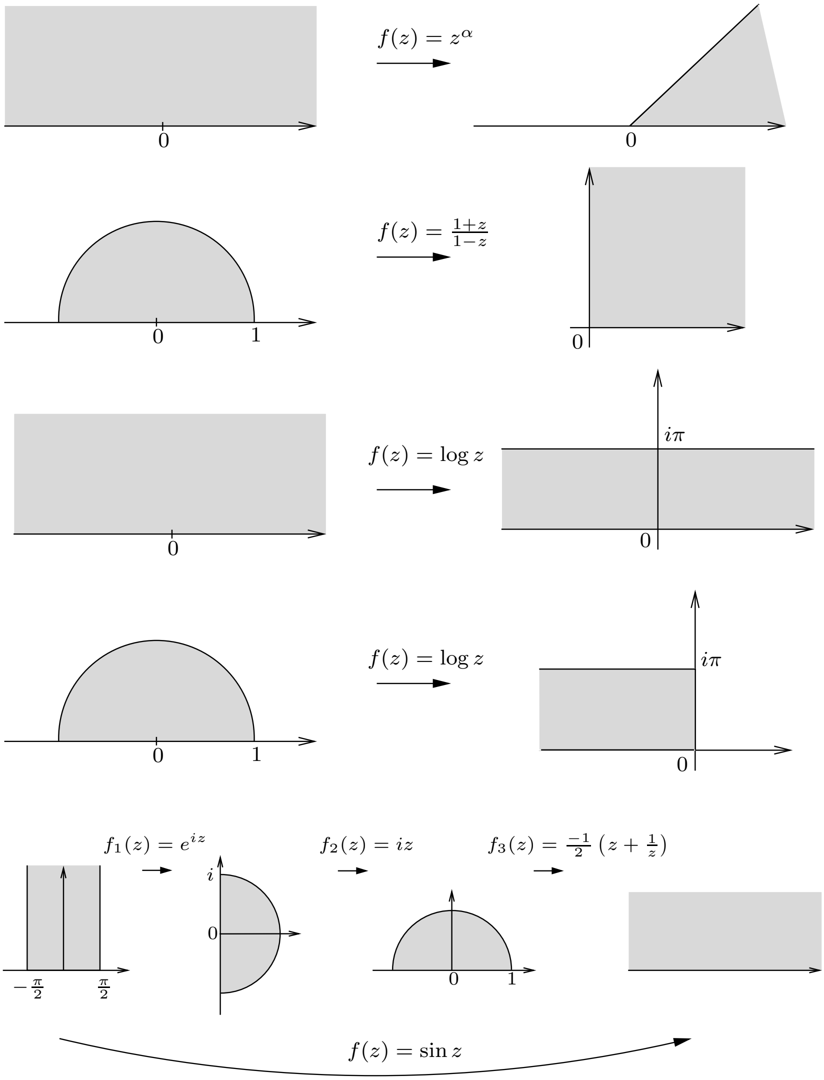

Figure 1. Explicit conformal maps

upper half-plane, where it arose in the solution of the steady-state heat equation. In these specific examples, explicit solutions were obtained in terms of convolutions with the Poisson kernels.

Our goal here is to connect the Dirichlet problem with the conformal maps discussed so far. We begin by providing a formula for a solution to the problem (1) in the special case where Ω is a strip. In fact, this example was studied in Problem 3 of Chapter 5, Book I, where the problem was solved using the Fourier transform. Here, we recover this solution using only conformal mappings and the known solution in the disc.

The first important fact that we use is that the composition of a harmonic function with a holomorphic function is still harmonic.

Lemma 1.3 *Let V and U be open sets in $\mathbb{C}$ and $F : V \to U$ a holomorphic function. If $u : U \to \mathbb{C}$ is a harmonic function, then $u \circ F$ is harmonic on V.*

Proof. The thrust of the lemma is purely local, so we may assume that U is an open disc. We let G be a holomorphic function in U whose real part is u (such a G exists by Exercise 12 in Chapter 2, and is determined up to an additive constant). Let $H = G \circ F$ and note that $u \circ F$ is the real part of H. Hence $u \circ F$ is harmonic because H is holomorphic.

For an alternate (computational) proof of this lemma, see Exercise 6.

With this result in hand, we may now consider the problem (1) when Ω consists of the horizontal strip

$$\Omega = \{x + iy : x \in \mathbb{R},\ 0 < y < 1\},$$

whose boundary is the union of the two horizontal lines $\mathbb{R}$ and $i + \mathbb{R}$. We express the boundary data as two functions f_0 and f_1 defined on $\mathbb{R}$, and ask for a solution $u(x, y)$ in Ω of $\triangle u = 0$ that satisfies

$$u(x, 0) = f_0(x) \quad \text{and} \quad u(x, 1) = f_1(x).$$

We shall assume that f_0 and f_1 are continuous and vanish at infinity, that is, that $\lim_{|x|\to\infty} f_j(x) = 0$ for $j = 0, 1$.

The method we shall follow consists of relocating the problem from the strip to the unit disc via a conformal map. In the disc the solution $\tilde{u}$ is then expressed in terms of a convolution with the Poisson kernel. Finally, $\tilde{u}$ is moved back to the strip using the inverse of the previous conformal map, thereby giving our final answer to the problem.

To achieve our goal, we introduce the mappings $F : \mathbb{D} \to \Omega$ and $G : \Omega \to \mathbb{D}$, that are defined by

$$F(w) = \frac{1}{\pi} \log\left(i\frac{1 - w}{1 + w} \right) \quad \text{and} \quad G(z) = \frac{i - e^{\pi z}}{i + e^{\pi z}}.$$

These two functions, which are obtained from composing mappings from examples in the previous sections, are conformal and inverses to one

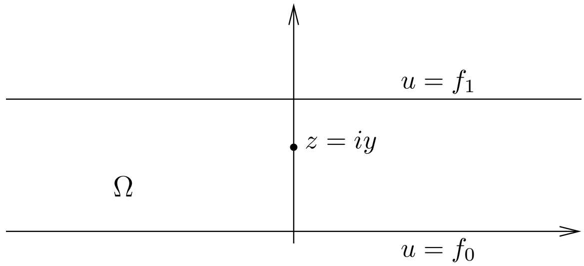

Figure 2. The Dirichlet problem in a strip

another. Tracing through the boundary behavior of F, we find that it maps the lower half-circle to the line $i + \mathbb{R}$, and the upper half-circle to $\mathbb{R}$. More precisely, as φ travels from $-\pi$ to 0, then $F(e^{i\varphi})$ goes from $i + \infty$ to $i - \infty$, and as φ travels from 0 to π, then $F(e^{i\varphi})$ goes from $-\infty$ to ∞ on the real line.

With the behavior of F on the circle in mind, we define

$$\tilde{f}_1(\varphi) = f_1(F(e^{i\varphi}) - i) \quad \text{whenever } -\pi < \varphi < 0,$$

and

$$\tilde{f}_0(\varphi) = f_0(F(e^{i\varphi})) \quad \text{whenever } 0 < \varphi < \pi.$$

Then, since f_0 and f_1 vanish at infinity, the function $\tilde{f}$ that is equal to $\tilde{f}_1$ on the lower semi-circle, $\tilde{f}_0$ on the upper semi-circle, and 0 at the points $\varphi = \pm\pi, 0$, is continuous on the whole circle. The solution to the Dirichlet problem in the unit disc with boundary data $\tilde{f}$ is given by the Poisson integral[3]

$$\begin{aligned}\tilde{u}(w) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - \varphi)\tilde{f}(\varphi)\,d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta - \varphi)\tilde{f}_1(\varphi)\,d\varphi + \frac{1}{2\pi}\int_{0}^{\pi} P_r(\theta - \varphi)\tilde{f}_0(\varphi)\,d\varphi,\end{aligned}$$

where $w = re^{i\theta}$, and

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

[3]We refer the reader to Chapter 2 in Book I for a detailed discussion of the Dirichlet problem in the disc and the Poisson integral formula. Also, the Poisson integral formula is deduced in Exercise 12 of Chapter 2 and Problem 2 in Chapter 3 of this book.

is the Poisson kernel. Lemma 1.3 guarantees that the function u, defined by

$$u(z) = \tilde{u}(G(z)),$$

is harmonic in the strip. Moreover, our construction also insures that u has the correct boundary values.

A formula for u in terms of f_0 and f_1 is first obtained at the points $z = iy$ with $0 < y < 1$. The appropriate change of variables (see Exercise 7) shows that if $re^{i\theta} = G(iy)$, then

$$\frac{1}{2\pi}\int_0^{\pi} P_r(\theta-\varphi)\tilde{f}_0(\varphi)\,d\varphi = \frac{\sin \pi y}{2}\int_{-\infty}^{\infty} \frac{f_0(t)}{\cosh \pi t - \cos \pi y}\,dt.$$

A similar calculation also establishes

$$\frac{1}{2\pi}\int_0^{\pi} P_r(\theta-\varphi)\tilde{f}_1(\varphi)\,d\varphi = \frac{\sin \pi y}{2}\int_{-\infty}^{\infty} \frac{f_1(t)}{\cosh \pi t + \cos \pi y}\,dt.$$

Adding these last two integrals provides a formula for $u(0, y)$. In general, we recall from Exercise 13 in Chapter 5 of Book I, that a solution to the Dirichlet problem in the strip vanishing at infinity is unique. Consequently, a translation of the boundary condition by x results in a translation of the solution by x as well. We may therefore apply the same argument to $f_0(x+t)$ and $f_1(x+t)$ (with x fixed), and a final change of variables shows that

$$u(x,y) = \frac{\sin \pi y}{2}\left(\int_{-\infty}^{\infty} \frac{f_0(x-t)}{\cosh \pi t - \cos \pi y}\,dt + \int_{-\infty}^{\infty} \frac{f_1(x-t)}{\cosh \pi t + \cos \pi y}\,dt\right),$$

which gives a solution to the Dirichlet problem in the strip. In particular, we find that the solution is given in terms of convolutions with the functions f_0 and f_1. Also, note that at the mid-point of the strip ($y = 1/2$), the solution is given by integration with respect to the function $1/\cosh \pi t$; this function happens to be its own Fourier transform, as we saw in Example 3, Chapter 3.

Remarks about the Dirichlet problem

The example above leads us to envisage the solution of the more general Dirichlet problem for Ω (a suitable region), if we know a conformal map F from the disc $\mathbb{D}$ to Ω. That is, suppose we wish to solve (1), where f is an assigned continuous function and $\partial\Omega$ is the boundary of Ω. Assuming we have a conformal map F from $\mathbb{D}$ to Ω (that extends to a continuous bijection of the boundary of the disc to the boundary of Ω), then $\tilde{f} = f \circ F$ is defined on the circle, and we can solve the Dirichlet problem for the disc with boundary data $\tilde{f}$. The solution is given by the Poisson integral formula

$$\tilde{u}(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) \tilde{f}(e^{i\varphi})\, d\varphi,$$

where P_r is the Poisson kernel. Then, one can expect that the solution of the original problem is given by $u = \tilde{u} \circ F^{-1}$.

Success with this approach requires that we are able to resolve affirmatively two questions:

- Does there exist a conformal map $\Phi = F^{-1}$ from Ω to $\mathbb{D}$?
- If so, does this map extend to a continuous bijection from the boundary of Ω to the boundary of $\mathbb{D}$?

The first question, that of existence, is settled by the Riemann mapping theorem, which we prove in the next section. It is completely general (assuming only that Ω is a proper subset of $\mathbb{C}$ that is simply connected), and necessitates no regularity of the boundary of Ω. A positive answer to the second question requires some regularity of $\partial\Omega$. A particular case, when Ω is the interior of a polygon, is treated below in Section 4.3. (See Exercise 18 and Problem 6 for more general assertions.)

It is interesting to note that in Riemann's original approach to the mapping problem, the chain of implications was reversed: his idea was that the existence of the conformal map Φ from Ω to $\mathbb{D}$ is a consequence of the solvability of the Dirichlet problem in Ω. He argued as follows. Suppose we wish to find such a Φ, with the property that a given point $z_0 \in \Omega$ is mapped to 0. Then Φ must be of the form

$$\Phi(z) = (z - z_0)G(z),$$

where G is holomorphic and non-vanishing in Ω. Hence we can take

$$\Phi(z) = (z - z_0)e^{H(z)},$$

for suitable H. Now if $u(z)$ is the harmonic function given by $u = \mathrm{Re}(H)$, then the fact that $|\Phi(z)| = 1$ on $\partial\Omega$ means that u must satisfy the boundary condition $u(z) = \log(1/|z - z_0|)$ for $z \in \partial\Omega$. So if we can find such a solution u of the Dirichlet problem,[4] we can construct H, and from this the mapping function Φ.

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[4]The harmonic function $u(z)$ is also known as the Green's function with source z_0 for the region Ω.

However, there are several shortcomings to this method. First, one has to verify that Φ is a bijection. In addition, to succeed, this method requires some regularity of the boundary of Ω. Moreover, one is still faced with the question of solving the Dirichlet problem for Ω. At this stage Riemann proposed using the "Dirichlet principle." But applying this idea involves difficulties that must be overcome.[5]

Nevertheless, using different methods, one can prove the existence of the mapping in the general case. This approach is carried out below in Section 3.

2 The Schwarz lemma; automorphisms of the disc and upper half-plane

The statement and proof of the Schwarz lemma are both simple, but the applications of this result are far-reaching. We recall that a rotation is a map of the form $z \mapsto cz$ with $|c| = 1$, namely $c = e^{i\theta}$, where $\theta \in \mathbb{R}$ is called the angle of rotation and is well-defined up to an integer multiple of 2π.

Lemma 2.1 *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then*

(i) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.

(ii) *If for some* $z_0 \neq 0$ *we have* $|f(z_0)| = |z_0|$*, then* f *is a rotation.*

(iii) $|f'(0)| \leq 1$*, and if equality holds, then* f *is a rotation.*

Proof. We first expand f in a power series centered at 0 and convergent in all of $\mathbb{D}$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots .$$

Since $f(0) = 0$ we have $a_0 = 0$, and therefore $f(z)/z$ is holomorphic in $\mathbb{D}$ (since it has a removable singularity at 0). If $|z| = r < 1$, then since $|f(z)| \leq 1$ we have

$$\left| \frac{f(z)}{z} \right| \leq \frac{1}{r},$$

and by the maximum modulus principle, we can conclude that this is true whenever $|z| \leq r$. Letting $r \to 1$ gives the first result.

For (ii), we see that $f(z)/z$ attains its maximum in the interior of $\mathbb{D}$ and must therefore be constant, say $f(z) = cz$. Evaluating this expression

[5] An implementation of Dirichlet's principle in the present two-dimensional situation is taken up in Book III.

at z_0 and taking absolute values, we find that $|c| = 1$. Therefore, there exists $\theta \in \mathbb{R}$ such that $c = e^{i\theta}$, and that explains why f is a rotation.

Finally, observe that if $g(z) = f(z)/z$, then $|g(z)| \le 1$ throughout $\mathbb{D}$, and moreover

$$g(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = f'(0).$$

Hence, if $|f'(0)| = 1$, then $|g(0)| = 1$, and by the maximum principle g is constant, which implies $f(z) = cz$ with $|c| = 1$.

Our first application of this lemma is to the determination of the automorphisms of the disc.

2.1 Automorphisms of the disc

A conformal map from an open set Ω to *itself* is called an **automorphism** of Ω. The set of all automorphisms of Ω is denoted by $\mathrm{Aut}(\Omega)$, and carries the structure of a group. The group operation is composition of maps, the identity element is the map $z \mapsto z$, and the inverses are simply the inverse functions. It is clear that if f and g are automorphisms of Ω, then $f \circ g$ is also an automorphism, and in fact, its inverse is given by

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$

As mentioned above, the identity map is always an automorphism. We can give other more interesting automorphisms of the unit disc. Obviously, any rotation by an angle $\theta \in \mathbb{R}$, that is, $r_\theta : z \mapsto e^{i\theta} z$, is an automorphism of the unit disc whose inverse is the rotation by the angle $-\theta$, that is, $r_{-\theta} : z \mapsto e^{-i\theta} z$. More interesting, are the automorphisms of the form

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1.$$

These mappings, which where introduced in Exercise 7 of Chapter 1, appear in a number of problems in complex analysis because of their many useful properties. The proof that they are automorphisms of $\mathbb{D}$ is quite simple. First, observe that since $|\alpha| < 1$, the map ψ_α is holomorphic in the unit disc. If $|z| = 1$ then $z = e^{i\theta}$ and

$$\psi_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \overline{\alpha})} = e^{-i\theta} \frac{w}{\overline{w}},$$

where $w = \alpha - e^{i\theta}$, therefore $|\psi_\alpha(z)| = 1$. By the maximum modulus principle, we conclude that $|\psi_\alpha(z)| < 1$ for all $z \in \mathbb{D}$. Finally we make the following very simple observation:

$$\begin{aligned}(\psi_\alpha \circ \psi_\alpha)(z) &= \frac{\alpha - \frac{\alpha - z}{1-\overline{\alpha}z}}{1 - \overline{\alpha}\frac{\alpha - z}{1-\overline{\alpha}z}} \\ &= \frac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \overline{\alpha}z - |\alpha|^2 + \overline{\alpha}z} \\ &= \frac{(1-|\alpha|^2)z}{1-|\alpha|^2} \\ &= z,\end{aligned}$$

from which we conclude that ψ_α is its own inverse! Another important property of ψ_α is that it vanishes at $z = \alpha$; moreover it interchanges 0 and α, namely

$$\psi_\alpha(0) = \alpha \quad \text{and} \quad \psi_\alpha(\alpha) = 0.$$

The next theorem says that the rotations combined with the maps ψ_α exhaust all the automorphisms of the disc.

Theorem 2.2 *If f is an automorphism of the disc, then there exist $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta}\frac{\alpha - z}{1 - \overline{\alpha}z}.$$

Proof. Since f is an automorphism of the disc, there exists a unique complex number $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$. Now we consider the automorphism g defined by $g = f \circ \psi_\alpha$. Then $g(0) = 0$, and the Schwarz lemma gives

$$|g(z)| \le |z| \quad \text{for all } z \in \mathbb{D}. \tag{2}$$

Moreover, $g^{-1}(0) = 0$, so applying the Schwarz lemma to g^{-1}, we find that

$$|g^{-1}(w)| \le |w| \quad \text{for all } w \in \mathbb{D}.$$

Using this last inequality for $w = g(z)$ for each $z \in \mathbb{D}$ gives

$$|z| \le |g(z)| \quad \text{for all } z \in \mathbb{D}. \tag{3}$$

Combining (2) and (3) we find that $|g(z)| = |z|$ for all $z \in \mathbb{D}$, and by the Schwarz lemma we conclude that $g(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$. Replacing z by $\psi_\alpha(z)$ and using the fact that $(\psi_\alpha \circ \psi_\alpha)(z) = z$, we deduce that $f(z) = e^{i\theta}\psi_\alpha(z)$, as claimed.

Setting $\alpha = 0$ in the theorem yields the following result.

Corollary 2.3 *The only automorphisms of the unit disc that fix the origin are the rotations.*

Note that by the use of the mappings ψ_α, we can see that the group of automorphisms of the disc acts **transitively**, in the sense that given any pair of points α and β in the disc, there is an automorphism ψ mapping α to β. One such ψ is given by $\psi = \psi_\beta \circ \psi_\alpha$.

The explicit formulas for the automorphisms of $\mathbb{D}$ give a good description of the group $\text{Aut}(\mathbb{D})$. In fact, this group of automorphisms is "almost" isomorphic to a group of 2×2 matrices with complex entries often denoted by $\text{SU}(1,1)$. This group consists of all 2×2 matrices that preserve the hermitian form on $\mathbb{C}^2 \times \mathbb{C}^2$ defined by

$$\langle Z, W \rangle = z_1 \overline{w}_1 - z_2 \overline{w}_2,$$

where $Z = (z_1, z_2)$ and $W = (w_1, w_2)$. For more information about this subject, we refer the reader to Problem 4.

2.2 Automorphisms of the upper half-plane

Our knowledge of the automorphisms of $\mathbb{D}$ together with the conformal map $F : \mathbb{H} \to \mathbb{D}$ found in Section 1.1 allow us to determine the group of automorphisms of $\mathbb{H}$ which we denote by $\text{Aut}(\mathbb{H})$.

Consider the map

$$\Gamma : \text{Aut}(\mathbb{D}) \to \text{Aut}(\mathbb{H})$$

given by "conjugation by F":

$$\Gamma(\varphi) = F^{-1} \circ \varphi \circ F.$$

It is clear that $\Gamma(\varphi)$ is an automorphism of $\mathbb{H}$ whenever φ is an automorphism of $\mathbb{D}$, and Γ is a bijection whose inverse is given by $\Gamma^{-1}(\psi) = F \circ \psi \circ F^{-1}$. In fact, we prove more, namely that Γ preserves the operations on the corresponding groups of automorphisms. Indeed, suppose that $\varphi_1, \varphi_2 \in \text{Aut}(\mathbb{D})$. Since $F \circ F^{-1}$ is the identity on $\mathbb{D}$ we find that

$$\begin{aligned}\Gamma(\varphi_1 \circ \varphi_2) &= F^{-1} \circ \varphi_1 \circ \varphi_2 \circ F \\ &= F^{-1} \circ \varphi_1 \circ F \circ F^{-1} \circ \varphi_2 \circ F \\ &= \Gamma(\varphi_1) \circ \Gamma(\varphi_2).\end{aligned}$$

The conclusion is that the two groups $\text{Aut}(\mathbb{D})$ and $\text{Aut}(\mathbb{H})$ are the same, since Γ defines an isomorphism between them. We are still left with the task of giving a description of elements of $\text{Aut}(\mathbb{H})$. A series of calculations, which consist of pulling back the automorphisms of the disc to the upper half-plane via F, can be used to verify that $\text{Aut}(\mathbb{H})$ consists of all maps

$$z \mapsto \frac{az+b}{cz+d},$$

where a, b, c, and d are real numbers with $ad - bc = 1$. Again, a matrix group is lurking in the background. Let $\text{SL}_2(\mathbb{R})$ denote the group of all 2×2 matrices with real entries and determinant 1, namely

$$\text{SL}_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad a, b, c, d \in \mathbb{R} \text{ and } \det(M) = ad - bc{=}1 \right\}.$$

This group is called the **special linear group**.

Given a matrix $M \in \text{SL}_2(\mathbb{R})$ we define the mapping f_M by

$$f_M(z) = \frac{az+b}{cz+d}.$$

Theorem 2.4 *Every automorphism of $\mathbb{H}$ takes the form f_M for some $M \in \text{SL}_2(\mathbb{R})$. Conversely, every map of this form is an automorphism of $\mathbb{H}$.*

The proof consists of a sequence of steps. For brevity, we denote the group $\text{SL}_2(\mathbb{R})$ by $\mathcal{G}$.

Step 1. If $M \in \mathcal{G}$, then f_M maps $\mathbb{H}$ to itself. This is clear from the observation that

$$(4) \quad \text{Im}(f_M(z)) = \frac{(ad-bc)\text{Im}(z)}{|cz+d|^2} = \frac{\text{Im}(z)}{|cz+d|^2} > 0 \quad \text{whenever } z \in \mathbb{H}.$$

Step 2. If M and M' are two matrices in $\mathcal{G}$, then $f_M \circ f_{M'} = f_{MM'}$. This follows from a straightforward calculation, which we omit. As a consequence, we can prove the first half of the theorem. Each f_M is an automorphism because it has a holomorphic inverse $(f_M)^{-1}$, which is simply $f_{M^{-1}}$. Indeed, if I is the identity matrix, then

$$(f_M \circ f_{M^{-1}})(z) = f_{MM^{-1}}(z) = f_I(z) = z.$$

Step 3. Given any two points z and w in $\mathbb{H}$, there exists $M \in \mathcal{G}$ such that $f_M(z) = w$, and therefore $\mathcal{G}$ acts transitively on $\mathbb{H}$. To prove this, it suffices to show that we can map any $z \in \mathbb{H}$ to i. Setting $d = 0$ in equation (4) above gives

$$\text{Im}(f_M(z)) = \frac{\text{Im}(z)}{|cz|^2}$$

and we may choose a real number c so that $\text{Im}(f_M(z)) = 1$. Next we choose the matrix

$$M_1 = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}$$

so that $f_{M_1}(z)$ has imaginary part equal to 1. Then we translate by a matrix of the form

$$M_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad \text{with } b \in \mathbb{R},$$

to bring $f_{M_1}(z)$ to i. Finally, the map f_M with $M = M_2 M_1$ takes z to i.

Step 4. If θ is real, then the matrix

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

belongs to $\mathcal{G}$, and if $F : \mathbb{H} \to \mathbb{D}$ denotes the standard conformal map, then $F \circ f_{M_\theta} \circ F^{-1}$ corresponds to the rotation of angle -2θ in the disc. This follows from the fact that $F \circ f_{M_\theta} = e^{-2i\theta} F(z)$, which is easily verified.

Step 5. We can now complete the proof of the theorem. We suppose f is an automorphism of $\mathbb{H}$ with $f(\beta) = i$, and consider a matrix $N \in \mathcal{G}$ such that $f_N(i) = \beta$. Then $g = f \circ f_N$ satisfies $g(i) = i$, and therefore $F \circ g \circ F^{-1}$ is an automorphism of the disc that fixes the origin. So $F \circ g \circ F^{-1}$ is a rotation, and by Step 4 there exists $\theta \in \mathbb{R}$ such that

$$F \circ g \circ F^{-1} = F \circ f_{M_\theta} \circ F^{-1}.$$

Hence $g = f_{M_\theta}$, and we conclude that $f = f_{M_\theta N^{-1}}$ which is of the desired form.

A final observation is that the group $\text{Aut}(\mathbb{H})$ is not quite isomorphic with $\text{SL}_2(\mathbb{R})$. The reason for this is because the two matrices M and $-M$ give rise to the same function $f_M = f_{-M}$. Therefore, if we identify the two matrices M and $-M$, then we obtain a new group $\text{PSL}_2(\mathbb{R})$ called the **projective special linear group**; this group is isomorphic with $\text{Aut}(\mathbb{H})$.

3 The Riemann mapping theorem

3.1 Necessary conditions and statement of the theorem

We now come to the promised cornerstone of this chapter. The basic problem is to determine conditions on an open set Ω that guarantee the existence of a conformal map $F : \Omega \to \mathbb{D}$.

A series of simple observations allow us to find necessary conditions on Ω. First, if $\Omega = \mathbb{C}$ there can be no conformal map $F : \Omega \to \mathbb{D}$, since by Liouville's theorem F would have to be a constant. Therefore, a necessary condition is to assume that $\Omega \neq \mathbb{C}$. Since $\mathbb{D}$ is connected, we must also impose the requirement that Ω be connected. There is still one more condition that is forced upon us: since $\mathbb{D}$ is simply connected, the same must be true of Ω (see Exercise 3). It is remarkable that these conditions on Ω are also sufficient to guarantee the existence of a biholomorpism from Ω to $\mathbb{D}$.

For brevity, we shall call a subset Ω of $\mathbb{C}$ **proper** if it is non-empty and not the whole of $\mathbb{C}$.

Theorem 3.1 (Riemann mapping theorem) *Suppose Ω is proper and simply connected. If $z_0 \in \Omega$, then there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that*

$$F(z_0) = 0 \quad \textit{and} \quad F'(z_0) > 0.$$

Corollary 3.2 *Any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.*

Clearly, the corollary follows from the theorem, since we can use as an intermediate step the unit disc. Also, the uniqueness statement in the theorem is straightforward, since if F and G are conformal maps from Ω to $\mathbb{D}$ that satisfy these two conditions, then $H = F \circ G^{-1}$ is an automorphism of the disc that fixes the origin. Therefore $H(z) = e^{i\theta} z$, and since $H'(0) > 0$, we must have $e^{i\theta} = 1$, from which we conclude that $F = G$.

The rest of this section is devoted to the proof of the existence of the conformal map F. The idea of the proof is as follows. We consider all injective holomorphic maps $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$. From these we wish to choose an f so that its image fills out all of $\mathbb{D}$, and this can be achieved by making $f'(z_0)$ as large as possible. In doing this, we shall need to be able to extract f as a limit from a given sequence of functions. We turn to this point first.

3.2 Montel's theorem

Let Ω be an open subset of $\mathbb{C}$. A family $\mathcal{F}$ of holomorphic functions on Ω is said to be **normal** if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of Ω (the limit need not be in $\mathcal{F}$).

The proof that a family of functions is normal is, in practice, the consequence of two related properties, uniform boundedness and equicontinuity. These we shall now define.

The family $\mathcal{F}$ is said to be **uniformly bounded on compact subsets of** Ω if for each compact set $K \subset \Omega$ there exists $B > 0$, such that

$$|f(z)| \leq B \quad \text{for all } z \in K \text{ and } f \in \mathcal{F}.$$

Also, the family $\mathcal{F}$ is **equicontinuous** on a compact set K if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $z, w \in K$ and $|z - w| < \delta$, then

$$|f(z) - f(w)| < \epsilon \quad \text{for all } f \in \mathcal{F}.$$

Equicontinuity is a strong condition, which requires uniform continuity, uniformly in the family. For instance, any family of differentiable functions on $[0,1]$ whose derivatives are uniformly bounded is equicontinuous. This follows directly from the mean value theorem. On the other hand, note that the family $\{f_n\}$ on $[0,1]$ given by $f_n(x) = x^n$ is *not* equicontinuous since for any fixed $0 < x_0 < 1$ we have $|f_n(1) - f_n(x_0)| \to 1$ as n tends to infinity.

The theorem that follows puts together these new concepts and is an important ingredient in the proof of the Riemann mapping theorem.

Theorem 3.3 *Suppose $\mathcal{F}$ is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω. Then:*

(i) *$\mathcal{F}$ is equicontinuous on every compact subset of Ω.*

(ii) *$\mathcal{F}$ is a normal family.*

The theorem really consists of two separate parts. The first part says that $\mathcal{F}$ is equicontinuous under the assumption that $\mathcal{F}$ is a family of *holomorphic* functions that is uniformly bounded on compact subsets of Ω. The proof follows from an application of the Cauchy integral formula and hence relies on the fact that $\mathcal{F}$ consists of holomorphic functions. This conclusion is in sharp contrast with the real situation as illustrated by the family of functions given by $f_n(x) = \sin(nx)$ on $(0,1)$, which is uniformly bounded. However, this family is not equicontinuous and has no convergent subsequence on any compact subinterval of $(0,1)$.

The second part of the theorem is not complex-analytic in nature. Indeed, the fact that $\mathcal{F}$ is a normal family follows from assuming only that $\mathcal{F}$ is uniformly bounded and equicontinuous on compact subsets of Ω. This result is sometimes known as the Arzela-Ascoli theorem and its proof consists primarily of a diagonalization argument.

We are required to prove convergence on arbitrary compact subsets of Ω, therefore it is useful to introduce the following notion. A sequence $\{K_\ell\}_{\ell=1}^{\infty}$ of compact subsets of Ω is called an **exhaustion** if

(a) K_ℓ is contained in the interior of $K_{\ell+1}$ for all $\ell = 1, 2, \ldots$.

(b) Any compact set $K \subset \Omega$ is contained in K_ℓ for some ℓ. In particular
$$\Omega = \bigcup_{\ell=1}^{\infty} K_\ell.$$

Lemma 3.4 *Any open set Ω in the complex plane has an exhaustion.*

Proof. If Ω is bounded, we let K_ℓ denote the set of all points in Ω at distance $\geq 1/\ell$ from the boundary of Ω. If Ω is not bounded, let K_ℓ denote the same set as above except that we also require $|z| \leq \ell$ for all $z \in K_\ell$.

We may now begin the proof of Montel's theorem. Let K be a compact subset of Ω and choose $r > 0$ so small that $D_{3r}(z)$ is contained in Ω for all $z \in K$. It suffices to choose r so that $3r$ is less than the distance from K to the boundary of Ω. Let $z, w \in K$ with $|z - w| < r$, and let γ denote the boundary circle of the disc $D_{2r}(w)$. Then, by Cauchy's integral formula, we have

$$f(z) - f(w) = \frac{1}{2\pi i} \int_\gamma f(\zeta) \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right] d\zeta.$$

Observe that

$$\left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| = \frac{|z - w|}{|\zeta - z|\,|\zeta - w|} \leq \frac{|z - w|}{r^2}$$

since $\zeta \in \gamma$ and $|z - w| < r$. Therefore

$$|f(z) - f(w)| \leq \frac{1}{2\pi} \frac{2\pi r}{r^2} B|z - w|,$$

where B denotes the uniform bound for the family $\mathcal{F}$ in the compact set consisting of all points in Ω at a distance $\leq 2r$ from K. Therefore $|f(z) - f(w)| < C|z - w|$, and this estimate is true for all $z, w \in K$ with $|z - w| < r$ and $f \in \mathcal{F}$; thus this family is equicontinuous, as was to be shown.

To prove the second part of the theorem, we argue as follows. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$ and K a compact subset of Ω. Choose a sequence of points $\{w_j\}_{j=1}^{\infty}$ that is dense in Ω. Since $\{f_n\}$ is uniformly bounded, there exists a subsequence $\{f_{n,1}\} = \{f_{1,1}, f_{2,1}, f_{3,1}, \ldots\}$ of $\{f_n\}$ such that $f_{n,1}(w_1)$ converges.

From $\{f_{n,1}\}$ we can extract a subsequence $\{f_{n,2}\} = \{f_{1,2}, f_{2,2}, f_{3,2}, \ldots\}$ so that $f_{n,2}(w_2)$ converges. We may continue this process, and extract a subsequence $\{f_{n,j}\}$ of $\{f_{n,j-1}\}$ such that $f_{n,j}(w_j)$ converges.

Finally, let $g_n = f_{n,n}$ and consider the diagonal subsequence $\{g_n\}$. By construction, $g_n(w_j)$ converges for each j, and we claim that equicontinuity implies that g_n converges uniformly on K. Given $\epsilon > 0$, choose δ as in the definition of equicontinuity, and note that for some J, the set K is contained in the union of the discs $D_\delta(w_1), \ldots, D_\delta(w_J)$. Pick N so large that if $n, m > N$, then

$$|g_m(w_j) - g_n(w_j)| < \epsilon \quad \text{for all } j = 1, \ldots, J.$$

So if $z \in K$, then $z \in D_\delta(w_j)$ for some $1 \leq j \leq J$. Therefore,

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| +$$

$$+ |g_m(w_j) - g_m(z)| < 3\epsilon$$

whenever $n, m > N$. Hence $\{g_n\}$ converges uniformly on K.

Finally, we need one more diagonalization argument to obtain a subsequence that converges uniformly on *every* compact subset of Ω. Let $K_1 \subset K_2 \subset \cdots \subset K_\ell \subset \cdots$ be an exhaustion of Ω, and suppose $\{g_{n,1}\}$ is a subsequence of the original sequence $\{f_n\}$ that converges uniformly on K_1. Extract from $\{g_{n,1}\}$ a subsequence $\{g_{n,2}\}$ that converges uniformly on K_2, and so on. Then, $\{g_{n,n}\}$ is a subsequence of $\{f_n\}$ that converges uniformly on every K_ℓ and since the K_ℓ exhaust Ω, the sequence $\{g_{n,n}\}$ converges uniformly on any compact subset of Ω, as was to be shown.

We need one further result before we can give the proof of the Riemann mapping theorem.

Proposition 3.5 *If Ω is a connected open subset of $\mathbb{C}$ and $\{f_n\}$ a sequence of injective holomorphic functions on Ω that converges uniformly* *on every compact subset of* Ω *to a holomorphic function* f*, then* f *is either injective or constant.*

Proof. We argue by contradiction and suppose that f is not injective, so there exist distinct complex numbers z_1 and z_2 in Ω such that $f(z_1) = f(z_2)$. Define a new sequence by $g_n(z) = f_n(z) - f_n(z_1)$, so that g_n has no other zero besides z_1, and the sequence $\{g_n\}$ converges uniformly on compact subsets of Ω to $g(z) = f(z) - f(z_1)$. If g is not identically zero, then z_2 is an isolated zero for g (because Ω is connected); therefore

$$1 = \frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)}\,d\zeta,$$

where γ is a small circle centered at z_2 chosen so that g does not vanish on γ or at any point of its interior besides z_2. Therefore, $1/g_n$ converges uniformly to $1/g$ on γ, and since $g_n' \to g'$ uniformly on γ we have

$$\frac{1}{2\pi i}\int_\gamma \frac{g_n'(\zeta)}{g_n(\zeta)}\,d\zeta \to \frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)}\,d\zeta.$$

But this is a contradiction since g_n has no zeros inside γ, and hence

$$\frac{1}{2\pi i}\int_\gamma \frac{g_n'(\zeta)}{g_n(\zeta)}\,d\zeta = 0 \qquad \text{for all } n\,.$$

3.3 Proof of the Riemann mapping theorem

Once we have established the technical results above, the rest of the proof of the Riemann mapping theorem is very elegant. It consists of three steps, which we isolate.

Step 1. Suppose that Ω is a simply connected proper open subset of $\mathbb{C}$. We claim that Ω is conformally equivalent to an open subset of the unit disc that contains the origin. Indeed, choose a complex number α that does not belong to Ω, (recall that Ω is proper), and observe that $z - \alpha$ never vanishes on the simply connected set Ω. Therefore, we can define a holomorphic function

$$f(z) = \log(z - \alpha)$$

with the desired properties of the logarithm. As a consequence one has, $e^{f(z)} = z - \alpha$, which proves in particular that f is injective. Pick a point $w \in \Omega$, and observe that

$$f(z) \neq f(w) + 2\pi i \qquad \text{for all } z \in \Omega$$

for otherwise, we exponentiate this relation to find that $z = w$, hence $f(z) = f(w)$, a contradiction. In fact, we claim that $f(z)$ stays strictly away from $f(w) + 2\pi i$, in the sense that there exists a disc centered at $f(w) + 2\pi i$ that contains no points of the image $f(\Omega)$. Otherwise, there exists a sequence $\{z_n\}$ in Ω such that $f(z_n) \to f(w) + 2\pi i$. We exponentiate this relation, and, since the exponential function is continuous, we must have $z_n \to w$. But this implies $f(z_n) \to f(w)$, which is a contradiction. Finally, consider the map

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi i)}.$$

Since f is injective, so is F, hence $F : \Omega \to F(\Omega)$ is a conformal map. Moreover, by our analysis, $F(\Omega)$ is bounded. We may therefore translate and rescale the function F in order to obtain a conformal map from Ω to an open subset of $\mathbb{D}$ that contains the origin.

Step 2. By the first step, we may assume that Ω is an open subset of $\mathbb{D}$ with $0 \in \Omega$. Consider the family $\mathcal{F}$ of all injective holomorphic functions on Ω that map into the unit disc and fix the origin:

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \text{ holomorphic, injective and } f(0) = 0\}.$$

First, note that $\mathcal{F}$ is non-empty since it contains the identity. Also, this family is uniformly bounded by construction, since all functions are required to map into the unit disc.

Now, we turn to the question of finding a function $f \in \mathcal{F}$ that maximizes $|f'(0)|$. First, observe that the quantities $|f'(0)|$ are uniformly bounded as f ranges in $\mathcal{F}$. This follows from the Cauchy inequality (Corollary 4.3 in Chapter 2) for f' applied to a small disc centered at the origin.

Next, we let

$$s = \sup_{f \in \mathcal{F}} |f'(0)|,$$

and we choose a sequence $\{f_n\} \subset \mathcal{F}$ such that $|f_n'(0)| \to s$ as $n \to \infty$. By Montel's theorem (Theorem 3.3), this sequence has a subsequence that converges uniformly on compact sets to a holomorphic function f on Ω. Since $s \geq 1$ (because $z \mapsto z$ belongs to $\mathcal{F}$), f is non-constant, hence injective, by Proposition 3.5. Also, by continuity we have $|f(z)| \leq 1$ for all $z \in \Omega$ and from the maximum modulus principle we see that $|f(z)| < 1$. Since we clearly have $f(0) = 0$, we conclude that $f \in \mathcal{F}$ with $|f'(0)| = s$.

Step 3. In this last step, we demonstrate that f is a conformal map from Ω to $\mathbb{D}$. Since f is already injective, it suffices to prove that f is also surjective. If this were not true, we could construct a function in $\mathcal{F}$ with derivative at 0 greater than s. Indeed, suppose there exists $\alpha \in \mathbb{D}$ such that $f(z) \neq \alpha$, and consider the automorphism ψ_α of the disc that interchanges 0 and α, namely

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Since Ω is simply connected, so is $U = (\psi_\alpha \circ f)(\Omega)$, and moreover, U does not contain the origin. It is therefore possible to define a square root function on U by

$$g(w) = e^{\frac{1}{2}\log w}.$$

Next, consider the function

$$F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f.$$

We claim that $F \in \mathcal{F}$. Clearly F is holomorphic and it maps 0 to 0. Also F maps into the unit disc since this is true of each of the functions in the composition. Finally, F is injective. This is clearly true for the automorphisms ψ_α and $\psi_{g(\alpha)}$; it is also true for the square root g and the function f, since the latter is injective by assumption. If h denotes the square function $h(w) = w^2$, then we must have

$$f = \psi_\alpha^{-1} \circ h \circ \psi_{g(\alpha)}^{-1} \circ F = \Phi \circ F.$$

But Φ maps $\mathbb{D}$ into $\mathbb{D}$ with $\Phi(0) = 0$, and is not injective because F is and h is not. By the last part of the Schwarz lemma, we conclude that $|\Phi'(0)| < 1$. The proof is complete once we observe that

$$f'(0) = \Phi'(0)F'(0),$$

and thus

$$|f'(0)| < |F'(0)|,$$

contradicting the maximality of $|f'(0)|$ in $\mathcal{F}$.

Finally, we multiply f by a complex number of absolute value 1 so that $f'(0) > 0$, which ends the proof.

For a variant of this proof, see Problem 7.

Remark. It is worthwhile to point out that the only places where the hypothesis of simple-connectivity entered in the proof were in the uses of the logarithm and the square root. Thus it would have sufficed to have assumed (in addition to the hypothesis that Ω is proper) that Ω is **holomorphically simply connected** in the sense that for any holomorphic function f in Ω and any closed curve γ in Ω, we have $\int_\gamma f(z)\,dz = 0$. Further discussion of this point, and various equivalent properties of simple-connectivity, are given in Appendix B.

4 Conformal mappings onto polygons

The Riemann mapping theorem guarantees the existence of a conformal map from any proper, simply connected open set to the disc, or equivalently to the upper half-plane, but this theorem gives little insight as to the exact form of this map. In Section 1 we gave various explicit formulas in the case of regions that have symmetries, but it is of course unreasonable to ask for an explicit formula in the general case. There is, however, another class of open sets for which there are nice formulas, namely the polygons. Our aim in this last section is to give a proof of the Schwarz-Christoffel formula, which describes the nature of conformal maps from the disc (or upper half-plane) to polygons.

4.1 Some examples

We begin by studying some motivating examples. The first two correspond to easy (but infinite and degenerate) cases.

Example 1. First, we investigate the conformal map from the upper half-plane to the sector $\{z : 0 < \arg z < \alpha\pi\}$, with $0 < \alpha < 2$, given in Section 1 by $f(z) = z^\alpha$. Anticipating the Schwarz-Christoffel formula below, we write

$$z^\alpha = f(z) = \int_0^z f'(\zeta)\,d\zeta = \alpha \int_0^z \zeta^{-\beta}\,d\zeta$$

with $\alpha + \beta = 1$, and where the integral is taken along any path in the upper half-plane. In fact, by continuity and Cauchy's theorem, we may take the path of integration to lie in the closure of the upper half-plane. Although the behavior of f follows immediately from the original definition, we study it in terms of the integral expression above, since this provides insight for the general case treated later.

Note first that $\zeta^{-\beta}$ is integrable near 0 since $\beta < 1$, therefore $f(0) = 0$. Observe that when z is real and positive $(z = x)$, then $f'(x) = \alpha x^{\alpha-1}$ is