Define the **Landau** kernels by

$$L_n(x) = \begin{cases} \dfrac{(1-x^2)^n}{c_n} & \text{if } -1 \le x \le 1, \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

where c_n is chosen so that $\int_{-\infty}^{\infty} L_n(x)\,dx = 1$. Prove that $\{L_n\}_{n\ge 0}$ is a family of good kernels as $n \to \infty$. As a result, show that if f is a continuous function supported in $[-1/2, 1/2]$, then $(f * L_n)(x)$ is a sequence of polynomials on $[-1/2, 1/2]$ which converges uniformly to f.

[Hint: First show that $c_n \ge 2/(n+1)$.]

11. Suppose that u is the solution to the heat equation given by $u = f * \mathcal{H}_t$ where $f \in \mathcal{S}(\mathbb{R})$. If we also set $u(x,0) = f(x)$, prove that u is continuous on the closure of the upper half-plane, and vanishes at infinity, that is,

$$u(x,t) \to 0 \quad \text{as } |x| + t \to \infty.$$

[Hint: To prove that u vanishes at infinity, show that (i) $|u(x,t)| \le C/\sqrt{t}$ and (ii) $|u(x,t)| \le C/(1+|x|^2) + Ct^{-1/2}e^{-cx^2/t}$. Use (i) when $|x| \le t$, and (ii) otherwise.]

12. Show that the function defined by

$$u(x,t) = \frac{x}{t}\mathcal{H}_t(x)$$

satisfies the heat equation for $t > 0$ and $\lim_{t\to 0} u(x,t) = 0$ for every x, but u is *not* continuous at the origin.

[Hint: Approach the origin with (x,t) on the parabola $x^2/4t = c$ where c is a constant.]

13. Prove the following uniqueness theorem for harmonic functions in the strip $\{(x,y) : 0 < y < 1,\ -\infty < x < \infty\}$: if u is harmonic in the strip, continuous on its closure with $u(x,0) = u(x,1) = 0$ for all $x \in \mathbb{R}$, and u vanishes at infinity, then $u = 0$.

14. Prove that the periodization of the Fejér kernel $\mathcal{F}_N$ on the real line (Exercise 9) is equal to the Fejér kernel for periodic functions of period 1. In other words,

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = F_N(x),$$

when $N \ge 1$ is an integer, and where

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = \frac{1}{N}\frac{\sin^2(N\pi x)}{\sin^2(\pi x)}.$$

shows that the only singularities $\text{sn}(z)$ are poles. Functions of this type, called "elliptic functions," are the subject of the next chapter.

5 Exercises

1. A holomorphic mapping $f : U \to V$ is a **local bijection** on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z, so that $f : D \to f(D)$ is a bijection.

Prove that a holomorphic map $f : U \to V$ is a local bijection on U if and only if $f'(z) \neq 0$ for all $z \in U$.

[Hint: Use Rouché's theorem as in the proof of Proposition 1.1.]

2. Supppose $F(z)$ is holomorphic near $z = z_0$ and $F(z_0) = F'(z_0) = 0$, while $F''(z_0) \neq 0$. Show that there are two curves Γ_1 and Γ_2 that pass through z_0, are orthogonal at z_0, and so that F restricted to Γ_1 is real and has a minimum at z_0, while F restricted to Γ_2 is also real but has a maximum at z_0.

[Hint: Write $F(z) = (g(z))^2$ for z near z_0, and consider the mapping $z \mapsto g(z)$ and its inverse.]

3. Suppose U and V are conformally equivalent. Prove that if U is simply connected, then so is V. Note that this conclusion remains valid if we merely assume that there exists a continuous bijection between U and V.

4. Does there exist a holomorphic surjection from the unit disc to $\mathbb{C}$?

[Hint: Move the upper half-plane "down" and then square it to get $\mathbb{C}$.]

5. Prove that $f(z) = -\frac{1}{2}(z + 1/z)$ is a conformal map from the half-disc $\{z = x + iy : |z| < 1,\ y > 0\}$ to the upper half-plane.

[Hint: The equation $f(z) = w$ reduces to the quadratic equation $z^2 + 2wz + 1 = 0$, which has two distinct roots in $\mathbb{C}$ whenever $w \neq \pm 1$. This is certainly the case if $w \in \mathbb{H}$.]

6. Give another proof of Lemma 1.3 by showing directly that the Laplacian of $u \circ F$ is zero.

[Hint: The real and imaginary parts of F satisfy the Cauchy-Riemann equations.]

7. Provide all the details in the proof of the formula for the solution of the Dirichlet problem in a strip discussed in Section 1.3. Recall that it suffices to compute the solution at the points $z = iy$ with $0 < y < 1$.

(a) Show that if $re^{i\theta} = G(iy)$, then

$$re^{i\theta} = i\frac{\cos \pi y}{1 + \sin \pi y}.$$

This leads to two separate cases: either $0 < y \leq 1/2$ and $\theta = \pi/2$, or $1/2 \leq$

$y < 1$ and $\theta = -\pi/2$. In either case, show that

$$r^2 = \frac{1 - \sin \pi y}{1 + \sin \pi y} \qquad \text{and} \qquad P_r(\theta - \varphi) = \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}.$$

(b) In the integral $\frac{1}{2\pi} \int_0^\pi P_r(\theta - \varphi) \tilde{f}_0(\varphi)\, d\varphi$ make the change of variables $t = F(e^{i\varphi})$. Observe that

$$e^{i\varphi} = \frac{i - e^{\pi t}}{i + e^{\pi t}},$$

and then take the imaginary part and differentiate both sides to establish the two identities

$$\sin \varphi = \frac{1}{\cosh \pi t} \qquad \text{and} \qquad \frac{d\varphi}{dt} = \frac{\pi}{\cosh \pi t}.$$

Hence deduce that

$$\begin{aligned} \frac{1}{2\pi} \int_0^\pi P_r(\theta - \varphi) \tilde{f}_0(\varphi)\, d\varphi &= \frac{1}{2\pi} \int_0^\pi \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi} \tilde{f}_0(\varphi)\, d\varphi \\ &= \frac{\sin \pi y}{2} \int_{-\infty}^{\infty} \frac{f_0(t)}{\cosh \pi t - \cos \pi y}\, dt. \end{aligned}$$

(c) Use a similar argument to prove the formula for the integral $\frac{1}{2\pi} \int_{-\pi}^0 P_r(\theta - \varphi) \tilde{f}_1(\varphi)\, d\varphi$.

8. Find a harmonic function u in the open first quadrant that extends continuously up to the boundary except at the points 0 and 1, and that takes on the following boundary values: $u(x, y) = 1$ on the half-lines $\{y = 0,\ x > 1\}$ and $\{x = 0,\ y > 0\}$, and $u(x, y) = 0$ on the segment $\{0 < x < 1, y = 0\}$.

[Hint: Find conformal maps $F_1, F_2, \ldots, F_5$ indicated in Figure 11. Note that $\frac{1}{\pi} \arg(z)$ is harmonic on the upper half-plane, equals 0 on the positive real axis, and 1 on the negative real axis.]

9. Prove that the function u defined by

$$u(x, y) = \text{Re}\left(\frac{i + z}{i - z}\right) \qquad \text{and} \qquad u(0, 1) = 0$$

is harmonic in the unit disc and vanishes on its boundary. Note that u is not bounded in $\mathbb{D}$.

10. Let $F : \mathbb{H} \to \mathbb{C}$ be a holomorphic function that satisfies

$$|F(z)| \leq 1 \qquad \text{and} \qquad F(i) = 0.$$