Übungsblatt 7

Fourieranalyse WS 2018/19

18.12.2018

Man bearbeite die folgenden Texte und trage in der Übungsgruppe darüber vor!

Let $f\colon (0, \infty) \to \mathbb{R}$ be continuous. The study of the behaviour $\sum_{r=1}^{n} \exp(2\pi i f(r))$ as $n \to \infty$ will be of great interest for numbe theory. This is the case, but we shall limit ourselves to a very simple exampl where f is slowly varying. (The arguments are due to Fejér.)

(i) Let $\phi(x) = x - [x] - 1/2$. Sketch the graph of ϕ.

(ii) Suppose f is differentiable. By integrating by parts, show that

$$\int_r^{r+1} \exp(2\pi i f(x))\, dx = 2^{-1}[\exp(2\pi i f(r+1)) + \exp(2\pi i f(r))]$$
$$- 2\pi i \int_r^{r+1} \phi(x) f'(x) \exp(2\pi i f(x))\, dx.$$

(iii) By summing the previous result, show that

$$\sum_{r=1}^{n} \exp(2\pi i f(r)) = 2^{-1}[\exp(2\pi i f(n)) + \exp(2\pi i f(1))]$$

$$+ \int_1^n \exp(2\pi i f(x))\, dx$$

$$+ 2\pi i \int_1^n \phi(x) f'(x) \exp(2\pi i f(x))\, dx.$$

(iv) Conclude that, if in addition $f'(x) \to 0$ as $x \to \infty$, then

$$n^{-1} \sum_{r=1}^{n} \exp(2\pi i f(r)) - n^{-1} \int_1^n \exp(2\pi i f(x))\, dx \to 0 \quad \text{as } n \to \infty.$$

(v) Now suppose that f satisfies the previous hypotheses and, in addition, $xf'(x) \to A$ as $x \to \infty$. Show that

$$\left| \frac{1}{n} \int_1^n x f'(x) \exp(2\pi i f(x))\, dx - \frac{A}{n} \int_1^n \exp(2\pi i f(x))\, dx \right| \to 0$$

as $n \to \infty$. By integrating $\int_1^n x f'(x) \exp(2\pi i f(x))\, dx$ by parts, show using (iv) that

$$n^{-1} \sum_{r=1}^{n} \exp(2\pi i f(r)) - \frac{\exp(2\pi i f(n))}{(1 + 2\pi i A)} \to 0$$

Conclude that $f(n)$ is not equidistributed.

$\langle ((1+\sqrt{5})/2)^n \rangle$ *is not equidistributed.*

Proof. By solving the difference equation or by straightforward induction it is easy to check that

$$u_r = \left(\frac{1+\sqrt{5}}{2}\right)^r + \left(\frac{1-\sqrt{5}}{2}\right)^r$$

satisfies the difference equation $u_{r+1} = u_r + u_{r-1}$, with the initial conditions $u_0 = 2$, $u_1 = 1$.

Thus u_r is always an integer. But $(1-\sqrt{5}/2)^r$ is negative for r odd, positive for r even and, more importantly, tends to 0 as $r \to \infty$ (since $0 > (1-\sqrt{5})/2 > -1$). Thus

$$\langle((1+\sqrt{5})/2)^{2r+1}\rangle \to 0 \quad \text{as } r \to \infty,$$
$$\langle((1+\sqrt{5})/2)^{2r}\rangle \to 1 \quad \text{as } r \to \infty,$$

and

$$\operatorname{card} \quad n^{-1}\{n \geqslant r \geqslant 1 : \langle((1+\sqrt{5})/2)^r\rangle \in [\tfrac{1}{4}, \tfrac{3}{4}]\} \to 0 \quad \text{as } n \to \infty. \qquad \blacksquare$$