7 The Zeta Function and Prime Number Theorem

> Bernhard Riemann, whose extraordinary intuitive powers we have already mentioned, has especially renovated our knowledge of the distribution of prime numbers, also one of the most mysterious questions in mathematics. He has taught us to deduce results in that line from considerations borrowed from the integral calculus: more precisely, from the study of a certain quantity, a function of a variable s which may assume not only real, but also imaginary values. He proved some important properties of that function, but enunciated two or three as important ones without giving the proof. At the death of Riemann, a note was found among his papers, saying "These properties of $\zeta(s)$ (the function in question) are deduced from an expression of it which, however, I did not succeed in simplifying enough to publish it."
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> We still have not the slightest idea of what the expression could be. As to the properties he simply enunciated, some thirty years elapsed before I was able to prove all of them but one. The question concerning that last one remains unsolved as yet, though, by an immense labor pursued throughout this last half century, some highly interesting discoveries in that direction have been achieved. It seems more and more probable, but still not at all certain, that the "Riemann hypothesis" is true.
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> *J. Hadamard,* 1945

Euler found, through his product formula for the zeta function, a deep connection between analytical methods and arithmetic properties of numbers, in particular primes. An easy consequence of Euler's formula is that the sum of the reciprocals of all primes, $\sum_p 1/p$, diverges, a result that quantifies the fact that there are infinitely many prime numbers. The natural problem then becomes that of understanding how these primes are distributed. With this in mind, we consider the following function:

$$\pi(x) = \text{number of primes less than or equal to } x.$$

The erratic growth of the function $\pi(x)$ gives little hope of finding a simple formula for it. Instead, one is led to study the asymptotic behavior of $\pi(x)$ as x becomes large. About 60 years after Euler's discovery, Legendre and Gauss observed after numerous calculations that it was likely that

$$(1) \qquad \pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$

(The asymptotic relation $f(x) \sim g(x)$ as $x \to \infty$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.) Another 60 years later, shortly before Riemann's work, Tchebychev proved by elementary methods (and in particular, without the zeta function) the weaker result that

$$(2) \qquad \pi(x) \approx \frac{x}{\log x} \quad \text{as } x \to \infty.$$

Here, by definition, the symbol $\approx$ means that there are positive constants $A < B$ such that

$$A \frac{x}{\log x} \leq \pi(x) \leq B \frac{x}{\log x}$$

for all sufficiently large x.

In 1896, about 40 years after Tchebychev's result, Hadamard and de la Vallée Poussin gave a proof of the validity of the relation (1). Their result is known as the prime number theorem. The original proofs of this theorem, as well as the one we give below, use complex analysis. We should remark that since then other proofs have been found, some depending on complex analysis, and others more elementary in nature.

At the heart of the proof of the prime number theorem that we give below lies the fact that $\zeta(s)$ does not vanish on the line $\text{Re}(s) = 1$. In fact, it can be shown that these two propositions are equivalent.

1 Zeros of the zeta function

We have seen in Theorem 1.10, Chapter 8 in Book I, Euler's identity, which states that for $\text{Re}(s) > 1$ the zeta function can be expressed as an infinite product

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

For the sake of completeness we provide a proof of the above identity. The key observation is that $1/(1-p^{-s})$ can be written as a convergent (geometric) power series

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}} + \cdots ,$$

and taking formally the product of these series over all primes p, yields the desired result. A precise argument goes as follows.

Suppose M and N are positive integers with $M > N$. Observe now that, by the fundamental theorem of arithmetic,[1] any positive integer $n \leq N$ can be written uniquely as a product of primes, and that each prime that occurs in the product must be less than or equal to N and repeated less than M times. Therefore

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n^s} &\leq \prod_{p \leq N} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}}\right) \\
&\leq \prod_{p \leq N} \left(\frac{1}{1-p^{-s}}\right) \\
&\leq \prod_{p} \left(\frac{1}{1-p^{-s}}\right).
\end{aligned}$$

Letting N tend to infinity in the series now yields

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \leq \prod_{p} \left(\frac{1}{1-p^{-s}}\right).$$

For the reverse inequality, we argue as follows. Again, by the fundamental theorem of arithmetic, we find that

$$\prod_{p \leq N} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Letting M tend to infinity gives

$$\prod_{p \leq N} \left(\frac{1}{1-p^{-s}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

[1] A proof of this elementary (but essential) fact is given in the first section of Chapter 8 in Book I.

Hence

$$\prod_p \left(\frac{1}{1-p^{-s}} \right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and the proof of the product formula for ζ is complete.

From the product formula we see, by Proposition 3.1 in Chapter 5, that $\zeta(s)$ does not vanish when $\text{Re}(s) > 1$.

To obtain further information about the location of the zeros of ζ, we use the functional equation that provided the analytic continuation of ζ. We may write the fundamental relation $\xi(s) = \xi(1-s)$ in the form

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),$$

and therefore

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s).$$

Now observe that for $\text{Re}(s) < 0$ the following are true:

(i) $\zeta(1-s)$ has no zeros because $\text{Re}(1-s) > 1$.

(ii) $\Gamma((1-s)/2)$ is zero free.

(iii) $1/\Gamma(s/2)$ has zeros at $s = -2, -4, -6, \ldots$.

Therefore, the only zeros of ζ in $\text{Re}(s) < 0$ are located at the negative even integers $-2, -4, -6, \ldots$.

This proves the following theorem.

Theorem 1.1 *The only zeros of* ζ *outside the strip* $0 \leq \text{Re}(s) \leq 1$ *are at the negative even integers,* $-2, -4, -6, \ldots$.

The region that remains to be studied is called the **critical strip**, $0 \leq \text{Re}(s) \leq 1$. A key fact in the proof of the prime number theorem is that ζ has no zeros on the line $\text{Re}(s) = 1$. As a simple consequence of this fact and the functional equation, it follows that ζ has no zeros on the line $\text{Re}(s) = 0$.

In the seminal paper where Riemann introduced the analytic continuation of the ζ function and proved its functional equation, he applied these insights to the theory of prime numbers, and wrote down "explicit" formulas for determining the distribution of primes. While he did not succeed in fully proving and exploiting his assertions, he did initiate many important new ideas. His analysis led him to believe the truth of what has since been called the **Riemann hypothesis**:

The zeros of $\zeta(s)$ *in the critical strip lie on the line* $\mathrm{Re}(s) = 1/2$.

He said about this: "It would certainly be desirable to have a rigorous demonstration of this proposition; nevertheless I have for the moment set this aside, after several quick but unsuccessful attempts, because it seemed unneeded for the immediate goal of my study." Although much of the theory and numerical results point to the validity of this hypothesis, a proof or a counter-example remains to be discovered. The Riemann hypothesis is today one of mathematics' most famous unresolved problems.

In particular, it is for this reason that the zeros of ζ located outside the critical strip are sometimes called the **trivial zeros** of the zeta function. See also Exercise 5 for an argument proving that ζ has no zeros on the real segment, $0 \le \sigma \le 1$, where $s = \sigma + it$.

In the rest of this section we shall restrict ourselves to proving the following theorem, together with related estimates on ζ, which we shall use in the proof of the prime number theorem.

Theorem 1.2 *The zeta function has no zeros on the line* $\mathrm{Re}(s) = 1$.

Of course, since we know that ζ has a pole at $s = 1$, there are no zeros in a neighborhood of this point, but what we need is the deeper property that

$$\zeta(1 + it) \neq 0 \qquad \text{for all } t \in \mathbb{R}.$$

The next sequence of lemmas gathers the necessary ingredients for the proof of Theorem 1.2.

Lemma 1.3 *If* $\mathrm{Re}(s) > 1$, *then*

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s}$$

for some $c_n \geq 0$.

Proof. Suppose first that $s > 1$. Taking the logarithm of the Euler product formula, and using the power series expansion for the logarithm

$$\log\left(\frac{1}{1-x}\right) = \sum_{m=1}^{\infty} \frac{x^m}{m},$$

which holds for $0 \le x < 1$, we find that

$$\log \zeta(s) = \log \prod_p \frac{1}{1-p^{-s}} = \sum_p \log\left(\frac{1}{1-p^{-s}}\right) = \sum_{p,m} \frac{p^{-ms}}{m}.$$

Since the double sum converges absolutely, we need not specify the order of summation. See the Note at the end of this chapter. The formula then holds for all $\text{Re}(s) > 1$ by analytic continuation. Note that, by Theorem 6.2 in Chapter 3, $\log \zeta(s)$ is well defined in the simply connected half-plane $\text{Re}(s) > 1$, since ζ has no zeros there. Finally, it is clear that we have

$$\sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s},$$

where $c_n = 1/m$ if $n = p^m$ and $c_n = 0$ otherwise.

The proof of the theorem we shall give depends on a simple trick that is based on the following inequality.

Lemma 1.4 *If $\theta \in \mathbb{R}$, then $3 + 4\cos\theta + \cos 2\theta \ge 0$.*

This follows at once from the simple observation

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2.$$

Corollary 1.5 *If $\sigma > 1$ and t is real, then*

$$\log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \ge 0.$$

Proof. Let $s = \sigma + it$ and note that

$$\text{Re}(n^{-s}) = \text{Re}(e^{-(\sigma+it)\log n}) = e^{-\sigma \log n}\cos(t\log n) = n^{-\sigma}\cos(t\log n).$$

Therefore,

$$\begin{aligned}
&\log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \\
&= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \\
&= 3\text{Re}[\log\zeta(\sigma)] + 4\text{Re}[\log\zeta(\sigma+it)] + \text{Re}[\log\zeta(\sigma+2it)] \\
&= \sum c_n n^{-\sigma}(3 + 4\cos\theta_n + \cos 2\theta_n),
\end{aligned}$$

where $\theta_n = t\log n$. The positivity now follows from Lemma 1.4, and the fact that $c_n \ge 0$.

We can now finish the proof of our theorem.

Proof of Theorem 1.2. Suppose on the contrary that $\zeta(1+it_0)=0$ for some $t_0 \neq 0$. Since ζ is holomorphic at $1+it_0$, it must vanish at least to order 1 at this point, hence

$$|\zeta(\sigma+it_0)|^4 \leq C(\sigma-1)^4 \quad \text{as } \sigma \to 1,$$

for some constant $C>0$. Also, we know that $s=1$ is a simple pole for $\zeta(s)$, so that

$$|\zeta(\sigma)|^3 \leq C'(\sigma-1)^{-3} \quad \text{as } \sigma \to 1,$$

for some constant $C'>0$. Finally, since ζ is holomorphic at the points $\sigma+2it_0$, the quantity $|\zeta(\sigma+2it_0)|$ remains bounded as $\sigma \to 1$. Putting these facts together yields

$$|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \to 0 \quad \text{as } \sigma \to 1\,,$$

which contradicts Corollary 1.5, since the logarithm of real numbers between 0 and 1 is negative. This concludes the proof that ζ is zero free on the real line $\text{Re}(s)=1$.

1.1 Estimates for $1/\zeta(s)$

The proof of the prime number theorem relies on detailed manipulations of the zeta function near the line $\text{Re}(s)=1$; the basic object involved is the logarithmic derivative $\zeta'(s)/\zeta(s)$. For this reason, besides the non-vanishing of ζ on the line, we need to know about the growth of ζ' and $1/\zeta$. The former was dealt with in Proposition 2.7 of Chapter 6; we now treat the latter.

The proposition that follows is actually a quantitative version of Theorem 1.2.

Proposition 1.6 *For every $\epsilon>0$, we have $1/|\zeta(s)| \leq c_\epsilon |t|^\epsilon$ when $s=\sigma+it$, $\sigma \geq 1$, and $|t| \geq 1$.*

Proof. From our previous observations, we clearly have that

$$|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 1, \quad \text{whenever } \sigma \geq 1.$$

Using the estimate for ζ in Proposition 2.7 of Chapter 6, we find that

$$|\zeta^4(\sigma+it)| \geq c|\zeta^{-3}(\sigma)|\,|t|^{-\epsilon} \geq c'(\sigma-1)^3|t|^{-\epsilon},$$

for all $\sigma \geq 1$ and $|t| \geq 1$. Thus

(3) $\quad |\zeta(\sigma + it)| \geq c'(\sigma - 1)^{3/4}|t|^{-\epsilon/4}, \qquad$ whenever $\sigma \geq 1$ and $|t| \geq 1$.

We now consider two separate cases, depending on whether the inequality $\sigma - 1 \geq A|t|^{-5\epsilon}$ holds, for some appropriate constant A (whose value we choose later).

If this inequality does hold, then (3) immediately provides

$$|\zeta(\sigma + it)| \geq A'|t|^{-4\epsilon},$$

and it suffices to replace 4ϵ by ϵ to conclude the proof of the desired estimate, in this case.

If, however, $\sigma - 1 < A|t|^{-5\epsilon}$, then we first select $\sigma' > \sigma$ with $\sigma' - 1 = A|t|^{-5\epsilon}$. The triangle inequality then implies

$$|\zeta(\sigma + it)| \geq |\zeta(\sigma' + it)| - |\zeta(\sigma' + it) - \zeta(\sigma + it)|,$$

and an application of the mean value theorem, together with the estimates for the derivative of ζ obtained in the previous chapter, give

$$|\zeta(\sigma' + it) - \zeta(\sigma + it)| \leq c''|\sigma' - \sigma|\,|t|^{\epsilon} \leq c''|\sigma' - 1|\,|t|^{\epsilon}.$$

These observations, together with an application of (3) where we set $\sigma = \sigma'$, show that

$$|\zeta(\sigma + it)| \geq c'(\sigma' - 1)^{3/4}|t|^{-\epsilon/4} - c''(\sigma' - 1)|t|^{\epsilon}.$$

Now choose $A = (c'/(2c''))^4$, and recall that $\sigma' - 1 = A|t|^{-5\epsilon}$. This gives precisely

$$c'(\sigma' - 1)^{3/4}|t|^{-\epsilon/4} = 2c''(\sigma' - 1)|t|^{\epsilon},$$

and therefore

$$|\zeta(\sigma + it)| \geq A''|t|^{-4\epsilon}.$$

On replacing 4ϵ by ϵ, the desired inequality is established, and the proof of the proposition is complete.

2 Reduction to the functions ψ and ψ_1

In his study of primes, Tchebychev introduced an auxiliary function whose behavior is to a large extent equivalent to the asymptotic distribution of primes, but which is easier to manipulate than $\pi(x)$. **Tchebychev's ψ-function** is defined by

$$\psi(x) = \sum_{p^m \leq x} \log p.$$

The sum is taken over those integers of the form p^m that are less than or equal to x. Here p is a prime number and m is a positive integer. There are two other formulations of ψ that we shall need. First, if we define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

then it is clear that

$$\psi(x) = \sum_{1 \leq n \leq x} \Lambda(n).$$

Also, it is immediate that

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p} \right] \log p$$

where $[u]$ denotes the greatest integer $\leq u$, and the sum is taken over the primes less than x. This formula follows from the fact that if $p^m \leq x$, then $m \leq \log x / \log p$.

The fact that $\psi(x)$ contains enough information about $\pi(x)$ to prove our theorem is given a precise meaning in the statement of the next proposition. In particular, this reduces the prime number theorem to a corresponding asymptotic statement about ψ.

Proposition 2.1 *If* $\psi(x) \sim x$ *as* $x \to \infty$, *then* $\pi(x) \sim x / \log x$ *as* $x \to \infty$.

Proof. The argument here is elementary. By definition, it suffices to prove the following two inequalities:

$$(4) \qquad 1 \leq \liminf_{x \to \infty} \, \pi(x) \frac{\log x}{x} \quad \text{and} \quad \limsup_{x \to \infty} \, \pi(x) \frac{\log x}{x} \leq 1.$$

To do so, first note that crude estimates give

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p} \right] \log p \leq \sum_{p \leq x} \frac{\log x}{\log p} \log p = \pi(x) \log x,$$

and dividing through by x yields

$$\frac{\psi(x)}{x} \leq \frac{\pi(x) \log x}{x}.$$

The asymptotic condition $\psi(x) \sim x$ implies the first inequality in (4). The proof of the second inequality is a little trickier. Fix $0 < \alpha < 1$, and note that

$$\psi(x) \geq \sum_{p \leq x} \log p \geq \sum_{x^\alpha < p \leq x} \log p \geq (\pi(x) - \pi(x^\alpha)) \log x^\alpha,$$

and therefore

$$\psi(x) + \alpha\pi(x^\alpha) \log x \geq \alpha\pi(x) \log x.$$

Dividing by x, noting that $\pi(x^\alpha) \leq x^\alpha$, $\alpha < 1$, and $\psi(x) \sim x$, gives

$$1 \geq \alpha \limsup_{x\to\infty} \pi(x) \frac{\log x}{x}.$$

Since $\alpha < 1$ was arbitrary, the proof is complete.

Remark. The converse of the proposition is also true: if $\pi(x) \sim x/\log x$ then $\psi(x) \sim x$. Since we shall not need this result, we leave the proof to the interested reader.

In fact, it will be more convenient to work with a close cousin of the ψ function. Define the function ψ_1 by

$$\psi_1(x) = \int_1^x \psi(u)\, du.$$

In the previous proposition we reduced the prime number theorem to the asymptotics of $\psi(x)$ as x tends to infinity. Next, we show that this follows from the asymptotics of ψ_1.

Proposition 2.2 *If $\psi_1(x) \sim x^2/2$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$, and therefore $\pi(x) \sim x/\log x$ as $x \to \infty$.*

Proof. By Proposition 2.1, it suffices to prove that $\psi(x) \sim x$ as $x \to \infty$. This will follow quite easily from the fact that if $\alpha < 1 < \beta$, then

$$\frac{1}{(1-\alpha)x} \int_{\alpha x}^x \psi(u)\, du \leq \psi(x) \leq \frac{1}{(\beta-1)x} \int_x^{\beta x} \psi(u)\, du.$$

The proof of this double inequality is immediate and relies simply on the fact that ψ is increasing. As a consequence, we find, for example, that

$$\psi(x) \leq \frac{1}{(\beta-1)x}[\psi_1(\beta x) - \psi_1(x)],$$

and therefore

$$\frac{\psi(x)}{x} \leq \frac{1}{(\beta-1)}\left[\frac{\psi_1(\beta x)}{(\beta x)^2}\beta^2 - \frac{\psi_1(x)}{x^2}\right].$$

In turn this implies

$$\limsup_{x\to\infty}\frac{\psi(x)}{x} \leq \frac{1}{\beta-1}\left[\frac{1}{2}\beta^2 - \frac{1}{2}\right] = \frac{1}{2}(\beta+1).$$

Since this result is true for all $\beta > 1$, we have proved that $\limsup_{x\to\infty}\psi(x)/x \leq 1$. A similar argument with $\alpha < 1$, then shows that $\liminf_{x\to\infty}\psi(x)/x \geq 1$, and the proof of the proposition is complete.

It is now time to relate ψ_1 (and therefore also ψ) and ζ. We proved in Lemma 1.3 that for $\mathrm{Re}(s) > 1$

$$\log\zeta(s) = \sum_{m,p}\frac{p^{-ms}}{m}.$$

Differentiating this expression gives

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{m,p}(\log p)p^{-ms} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.$$

We record this formula for $\mathrm{Re}(s) > 1$ as

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}. \tag{5}$$

The asymptotic behavior $\psi_1(x) \sim x^2/2$ will be a consequence via (5) of the relationship between ψ_1 and ζ, which is expressed by the following noteworthy integral formula.

Proposition 2.3 *For all $c > 1$*

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds. \tag{6}$$

To make the proof of this formula clear, we isolate the necessary contour integrals in a lemma.

Lemma 2.4 *If* $c > 0$, *then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}\,ds = \begin{cases} 0 & \text{if } 0 < a \leq 1, \\ 1 - 1/a & \text{if } 1 \leq a. \end{cases}$$

Here, the integral is over the vertical line $\text{Re}(s) = c$.

Proof. First note that since $|a^s| = a^c$, the integral converges. We suppose first that $1 \leq a$, and write $a = e^{\beta}$ with $\beta = \log a \geq 0$. Let

$$f(s) = \frac{a^s}{s(s+1)} = \frac{e^{s\beta}}{s(s+1)}.$$

Then $\text{res}_{s=0} f = 1$ and $\text{res}_{s=-1} f = -1/a$. For $T > 0$, consider the path $\Gamma(T)$ shown on Figure 1.

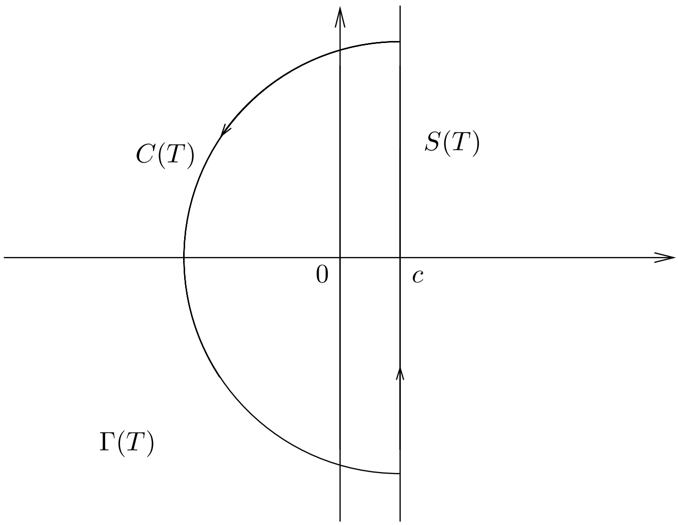

Figure 1. The contour in the proof of Lemma 2.4 when $a \geq 1$

The path $\Gamma(T)$ consists of the vertical segment $S(T)$ from $c - iT$ to $c + iT$, and of the half-circle $C(T)$ centered at c of radius T, lying to the left of the vertical segment. We equip $\Gamma(T)$ with the positive (counterclockwise) orientation, and note that we are dealing with a toy contour. If we choose T so large that 0 and -1 are contained in the interior of $\Gamma(T)$, then by the residue formula

$$\frac{1}{2\pi i}\int_{\Gamma(T)} f(s)\,ds = 1 - 1/a.$$

Since

$$\int_{\Gamma(T)} f(s)\,ds = \int_{S(T)} f(s)\,ds + \int_{C(T)} f(s)\,ds,$$

it suffices to prove that the integral over the half-circle goes to 0 as T tends to infinity. Note that if $s = \sigma + it \in C(T)$, then for all large T we have

$$|s(s+1)| \geq (1/2)T^2,$$

and since $\sigma \leq c$ we also have the estimate $|e^{\beta s}| \leq e^{\beta c}$. Therefore

$$\left|\int_{C(T)} f(s)\,ds\right| \leq \frac{C}{T^2} 2\pi T \to 0 \qquad \text{as } T \to \infty,$$

and the case when $a \geq 1$ is proved.

If $0 < a \leq 1$, consider an analogous contour but with the half-circle lying to the right of the line $\mathrm{Re}(s) = c$. Noting that there are no poles in the interior of that contour, we can give an argument similar to the one given above to show that the integral over the half-circle also goes to 0 as T tends to infinity.

We are now ready to prove Proposition 2.3. First, observe that

$$\psi(u) = \sum_{n=1}^{\infty} \Lambda(n) f_n(u),$$

where $f_n(u) = 1$ if $n \leq u$ and $f_n(u) = 0$ otherwise. Therefore,

$$\begin{aligned}
\psi_1(x) &= \int_0^x \psi(u)\,du \\
&= \sum_{n=1}^{\infty} \int_0^x \Lambda(n) f_n(u)\,du \\
&= \sum_{n \leq x} \Lambda(n) \int_n^x du,
\end{aligned}$$

and hence

$$\psi_1(x) = \sum_{n \leq x} \Lambda(n)(x-n).$$

This fact, together with equation (5) and an application of Lemma 2.4 (with $a = x/n$), gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds &= x\sum_{n=1}^{\infty}\Lambda(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s}{s(s+1)}\,ds\\ &= x\sum_{n\le x}\Lambda(n)\left(1-\frac{n}{x}\right)\\ &= \psi_1(x),\end{aligned}$$

as was to be shown.

2.1 Proof of the asymptotics for ψ_1

In this section, we will show that

$$\psi_1(x) \sim x^2/2 \qquad \text{as } x \to \infty,$$

and as a consequence, we will have proved the prime number theorem.

The key ingredients in the argument are:

- the formula in Proposition 2.3 connecting ψ_1 to ζ, namely

  $$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds$$

  for $c > 1$.

- the non-vanishing of the zeta function on $\mathrm{Re}(s) = 1$,

  $$\zeta(1+it) \neq 0 \qquad \text{for all } t \in \mathbb{R},$$

  and the estimates for ζ near that line given in Proposition 2.7 of Chapter 6 together with Proposition 1.6 of this chapter.

Let us now discuss our strategy in more detail. In the integral above for $\psi_1(x)$ we want to change the line of integration $\mathrm{Re}(s) = c$ with $c > 1$, to $\mathrm{Re}(s) = 1$. If we could achieve that, the size of the factor x^{s+1} in the integrand would then be of order x^2 (which is close to what we want) instead of x^{c+1}, $c > 1$, which is much too large. However, there would still be two issues that must be dealt with. The first is the pole of $\zeta(s)$ at $s = 1$; it turns out that when it is taken into account, its contribution is exactly the main term $x^2/2$ of the asymptotic of $\psi_1(x)$. Second, what remains must be shown to be essentially smaller than this term, and so we must further refine the crude estimate of order x^2 when integrating on the line $\mathrm{Re}(s) = 1$. We carry out our plan as follows.

Fix $c > 1$, say $c = 2$, and assume x is also fixed for the moment with $x \geq 2$. Let $F(s)$ denote the integrand

$$F(s) = \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right).$$

First we deform the vertical line from $c - i\infty$ to $c + i\infty$ to the path $\gamma(T)$ shown in Figure 2. (The segments of $\gamma(T)$ on the line $\mathrm{Re}(s) = 1$ consist of $T \leq t < \infty$, and $-\infty < t \leq -T$.) Here $T \geq 3$, and T will be chosen appropriately large later.

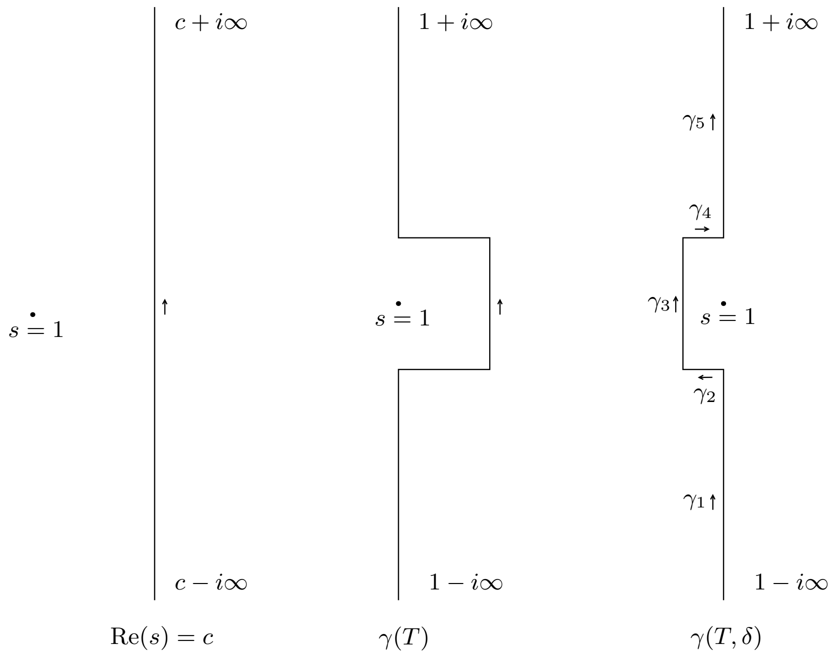

Figure 2. Three stages: the line $\mathrm{Re}(s) = c$, the contours $\gamma(T)$ and $\gamma(T, \delta)$

The usual and familiar arguments using Cauchy's theorem allow us to see that

$$\text{(7)} \qquad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, ds = \frac{1}{2\pi i} \int_{\gamma(T)} F(s)\, ds.$$

Indeed, we know on the basis of Proposition 2.7 in Chapter 6 and Proposition 1.6 that $|\zeta'(s)/\zeta(s)| \le A|t|^\eta$ for any fixed $\eta > 0$, whenever $s = \sigma + it$, $\sigma \ge 1$, and $|t| \ge 1$. Thus $|F(s)| \le A'|t|^{-2+\eta}$ in the two (infinite) rectangles bounded by the line $(c - i\infty, c + i\infty)$ and $\gamma(T)$. Since F is regular in that region, and its decrease at infinity is rapid enough, the assertion (7) is established.

Next, we pass from the contour $\gamma(T)$ to the contour $\gamma(T, \delta)$. (Again, see Figure 2.) For fixed T, we choose $\delta > 0$ small enough so that ζ has no zeros in the box

$$\{s = \sigma + it,\ 1 - \delta \le \sigma \le 1,\ |t| \le T\}.$$

Such a choice can be made since ζ does not vanish on the line $\sigma = 1$.

Now $F(s)$ has a simple pole at $s = 1$. In fact, by Corollary 2.6 in Chapter 6, we know that $\zeta(s) = 1/(s-1) + H(s)$, where $H(s)$ is regular near $s = 1$. Hence $-\zeta'(s)/\zeta(s) = 1/(s-1) + h(s)$, where $h(s)$ is holomorphic near $s = 1$, and so the residue of $F(s)$ at $s = 1$ equals $x^2/2$. As a result

$$\frac{1}{2\pi i}\int_{\gamma(T)} F(s)\,ds = \frac{x^2}{2} + \frac{1}{2\pi i}\int_{\gamma(T,\delta)} \frac{x^{s+1}}{s(s+1)}F(s)\,ds.$$

We now decompose the contour $\gamma(T, \delta)$ as $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$ and estimate each of the integrals $\int_{\gamma_j} F(s)\,ds$, $j = 1, 2, 3, 4, 5$, with the γ_j as in Figure 2.

First we contend that there exists T so large that

$$\left|\int_{\gamma_1} F(s)\,ds\right| \le \frac{\epsilon}{2}x^2 \quad \text{and} \quad \left|\int_{\gamma_5} F(s)\,ds\right| \le \frac{\epsilon}{2}x^2.$$

To see this, we first note that for $s \in \gamma_1$ one has

$$|x^{1+s}| = x^{1+\sigma} = x^2.$$

Then, by Proposition 1.6 we have, for example, that $|\zeta'(s)/\zeta(s)| \le A|t|^{1/2}$, so

$$\left|\int_{\gamma_1} F(s)\,ds\right| \le Cx^2 \int_T^\infty \frac{|t|^{1/2}}{t^2}\,dt.$$

Since the integral converges, we can make the right-hand side $\le \epsilon x^2/2$ upon taking T sufficiently large. The argument for the integral over γ_5 is the same.

Having now fixed T, we choose δ appropriately small. On γ_3, note that

$$|x^{1+s}| = x^{1+1-\delta} = x^{2-\delta}\,,$$

from which we conclude that there exists a constant C_T (dependent on T) such that

$$\left|\int_{\gamma_3} F(s)\,ds\right| \leq C_T x^{2-\delta}\,.$$

Finally, on the small horizontal segment γ_2 (and similarly on γ_4), we can estimate the integral as follows:

$$\left|\int_{\gamma_2} F(s)\,ds\right| \leq C_T' \int_{1-\delta}^{1} x^{1+\sigma}\,d\sigma \leq C_T' \frac{x^2}{\log x}\,.$$

We conclude that there exist constants C_T and C_T' (possibly different from the ones above) such that

$$\left|\psi_1(x) - \frac{x^2}{2}\right| \leq \epsilon x^2 + C_T x^{2-\delta} + C_T' \frac{x^2}{\log x}.$$

Dividing through by $x^2/2$, we see that

$$\left|\frac{2\psi_1(x)}{x^2} - 1\right| \leq 2\epsilon + 2C_T x^{-\delta} + 2C_T' \frac{1}{\log x},$$

and therefore, for all large x we have

$$\left|\frac{2\psi_1(x)}{x^2} - 1\right| \leq 4\epsilon.$$

This concludes the proof that

$$\psi_1(x) \sim x^2/2 \quad \text{ as } x \to \infty\,,$$

and thus, we have also completed the proof of the prime number theorem.

Note on interchanging double sums

We prove the following facts about the interchange of infinite sums: if $\{a_{k\ell}\}_{1\leq k,\ell<\infty}$ is a sequence of complex numbers indexed by $\mathbb{N}\times\mathbb{N}$, such that

(8) $$\sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{\infty}|a_{k\ell}|\right) < \infty,$$

then:

(i) The double sum $A = \sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{\infty} a_{k\ell}\right)$ summed in this order converges, and we may in fact also interchange the order of summation, so that

$$A = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} a_{k\ell} = \sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty} a_{k\ell}.$$

(ii) Given $\epsilon > 0$, there is a positive integer N so that for all $K, L > N$ we have $\left|A - \sum_{k=1}^{K}\sum_{\ell=1}^{L} a_{k\ell}\right| < \epsilon$.

(iii) If $m \mapsto (k(m), \ell(m))$ is a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, and if we write $c_m = a_{k(m)\ell(m)}$, then $A = \sum_{k=1}^{\infty} c_k$.

Statement (iii) says that any rearrangement of the sequence $\{a_{k\ell}\}$ can be summed without changing the limit. This is analogous to the case of absolutely convergent series, which can be summed in any desired order.

The condition (8) says that each sum $\sum_{\ell} a_{k\ell}$ converges absolutely, and moreover this convergence is "uniform" in k. An analogous situation arises for sequences of functions, where an important question is whether or not the interchange of limits

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) \stackrel{?}{=} \lim_{n\to\infty}\lim_{x\to x_0} f_n(x)$$

holds. It is a well-known fact that if the f_n's are continuous, and their convergence is uniform, then the above identity is true since the limit function is itself continuous. To take advantage of this fact, define $b_k = \sum_{\ell=1}^{\infty} |a_{k\ell}|$ and let $S = \{x_0, x_1, \ldots\}$ be a countable set of points with $\lim_{n\to\infty} x_n = x_0$. Also, define functions on S as follows:

$$\begin{aligned} f_k(x_0) &= \textstyle\sum_{\ell=1}^{\infty} a_{k\ell} && \text{for } k = 1, 2, \ldots \\ f_k(x_n) &= \textstyle\sum_{\ell=1}^{n} a_{k\ell} && \text{for } k = 1, 2, \ldots \text{ and } n = 1, 2, \ldots \\ g(x) &= \textstyle\sum_{k=1}^{\infty} f_k(x) && \text{for } x \in S. \end{aligned}$$

By assumption (8), each f_k is continuous at x_0. Moreover $|f_k(x)| \le b_k$ and $\sum b_k < \infty$, so the series defining the function g is uniformly convergent on S, and therefore g is also continuous at x_0. As a consequence we find (i), since

$$\begin{aligned} \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} a_{k\ell} = g(x_0) &= \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty}\sum_{k=1}^{\infty}\sum_{\ell=1}^{n} a_{k\ell} \\ &= \lim_{n\to\infty}\sum_{\ell=1}^{n}\sum_{k=1}^{\infty} a_{k\ell} = \sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty} a_{k\ell}. \end{aligned}$$

For the second statement, first observe that

$$\left|A - \sum_{k=1}^{K}\sum_{\ell=1}^{L} a_{k\ell}\right| \le \sum_{k\le K}\sum_{\ell>L} |a_{k\ell}| + \sum_{k>K}\sum_{\ell=1}^{\infty} |a_{k\ell}|.$$

To estimate the second term, we use the fact that $\sum b_k$ converges, which implies $\sum_{k>K}\sum_{\ell=1}^{\infty}|a_{k\ell}| < \epsilon/2$ whenever $K > K_0$, for some K_0. For the first term above, note that $\sum_{k\leq K}\sum_{\ell>L}|a_{k\ell}| \leq \sum_{k=1}^{\infty}\sum_{\ell>L}|a_{k\ell}|$. But the argument above guarantees that we can interchange these last two sums; also $\sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty}|a_{k\ell}| < \infty$, so that for all $L > L_0$ we have $\sum_{\ell>L}\sum_{k=1}^{\infty}|a_{k\ell}| < \epsilon/2$. Taking $N > \max(L_0, K_0)$ completes the proof of (ii).

The proof of (iii) is a direct consequence of (ii). Indeed, given any rectangle

$$R(K,L) = \{(k,\ell) \in \mathbb{N}\times\mathbb{N} : 1 \leq k \leq K \text{ and } 1 \leq \ell \leq L\},$$

there exists M such that the image of $[1, M]$ under the map $m \mapsto (k(m), \ell(m))$ contains $R(K,L)$.

When U denotes any open set in $\mathbb{R}^2$ that contains the origin, we define for $R > 0$ its dilate $U(R) = \{y \in \mathbb{R}^2 : y = Rx \text{ for some } x \in U\}$, and we can apply (ii) to see that

$$A = \lim_{R\to\infty} \sum_{(k,\ell)\in U(R)} a_{k\ell}.$$

In other words, under condition (8) the double sum $\sum_{k\ell} a_{k\ell}$ can be evaluated by summing over discs, squares, rectangles, ellipses, etc.

Finally, we leave the reader with the instructive task of finding a sequence of complex numbers $\{a_{k\ell}\}$ such that

$$\sum_k\sum_\ell a_{k\ell} \neq \sum_\ell\sum_k a_{k\ell}.$$

[Hint: Consider $\{a_{k\ell}\}$ as the entries of an infinite matrix with 0 above the diagonal, -1 on the diagonal, and $a_{k\ell} = 2^{\ell-k}$ if $k > \ell$.]

3 Exercises

1. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that the partial sums

$$A_n = a_1 + \cdots + a_n$$

are bounded. Prove that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in this half-plane.

[Hint: Use summation by parts to compare the original (non-absolutely convergent) series to the (absolutely convergent) series $\sum A_n(n^{-s} - (n+1)^{-s})$. An estimate for the term in parentheses is provided by the mean value theorem. To prove that the series is analytic, show that the partial sums converge uniformly on every compact subset of the half-plane $\mathrm{Re}(s) > 0$.]