for given z we can fix a choice of θ, and if z varies only a little, this determines the corresponding choice of θ uniquely (assuming we require that θ varies continuously with z). Thus "locally" we can give an unambiguous definition of the logarithm, but this will not work "globally." For example, if z starts at 1, and then winds around the origin and returns to 1, the logarithm does not return to its original value, but rather differs by an integer multiple of $2\pi i$, and therefore is not "single-valued." To make sense of the logarithm as a single-valued function, we must restrict the set on which we define it. This is the so-called choice of a **branch** or **sheet** of the logarithm.

Our discussion of simply connected domains given above leads to a natural global definition of a branch of the logarithm function.

Theorem 6.1 *Suppose that* Ω *is simply connected with* $1 \in \Omega$*, and* $0 \notin \Omega$*. Then in* Ω *there is a branch of the logarithm* $F(z) = \log_\Omega(z)$ *so that*

(i) F *is holomorphic in* Ω*,*

(ii) $e^{F(z)} = z$ *for all* $z \in \Omega$*,*

(iii) $F(r) = \log r$ *whenever* r *is a real number and near* 1*.*

In other words, each branch $\log_\Omega(z)$ is an extension of the standard logarithm defined for positive numbers.

Proof. We shall construct F as a primitive of the function $1/z$. Since $0 \notin \Omega$, the function $f(z) = 1/z$ is holomorphic in Ω. We define

$$\log_\Omega(z) = F(z) = \int_\gamma f(w)\, dw\,,$$

where γ is any curve in Ω connecting 1 to z. Since Ω is simply connected, this definition does not depend on the path chosen. Arguing as in the proof of Theorem 5.2, we find that F is holomorphic and $F'(z) = 1/z$ for all $z \in \Omega$. This proves (i). To prove (ii), it suffices to show that $ze^{-F(z)} = 1$. For that, we differentiate the left-hand side, obtaining

$$\frac{d}{dz}\left(ze^{-F(z)}\right) = e^{-F(z)} - zF'(z)e^{-F(z)} = (1 - zF'(z))e^{-F(z)} = 0.$$

Since Ω is connected we conclude, by Corollary 3.4 in Chapter 1, that $ze^{-F(z)}$ is constant. Evaluating this expression at $z = 1$, and noting that $F(1) = 0$, we find that this constant must be 1.

Finally, if r is real and close to 1 we can choose as a path from 1 to r a line segment on the real axis so that

$$F(r) = \int_1^r \frac{dx}{x} = \log r,$$

by the usual formula for the standard logarithm. This completes the proof of the theorem.

For example, in the slit plane $\Omega = \mathbb{C} - \{(-\infty, 0]\}$ we have the **principal branch** of the logarithm

$$\log z = \log r + i\theta$$

where $z = re^{i\theta}$ with $|\theta| < \pi$. (Here we drop the subscript Ω, and write simply $\log z$.) To prove this, we use the path of integration γ shown in Figure 8.

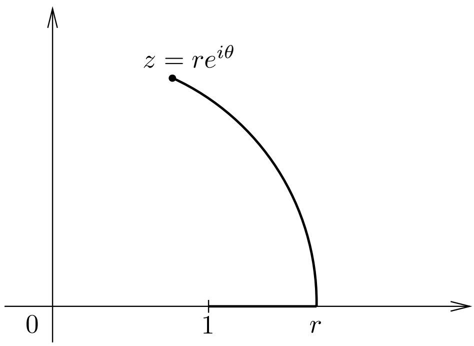

Figure 8. Path of integration for the principal branch of the logarithm

If $z = re^{i\theta}$ with $|\theta| < \pi$, the path consists of the line segment from 1 to r and the arc η from r to z. Then

$$\begin{aligned}\log z &= \int_1^r \frac{dx}{x} + \int_\eta \frac{dw}{w} \\ &= \log r + \int_0^\theta \frac{ire^{it}}{re^{it}}\, dt \\ &= \log r + i\theta.\end{aligned}$$

An important observation is that in general

$$\log(z_1 z_2) \neq \log z_1 + \log z_2.$$

For example, if $z_1 = e^{2\pi i/3} = z_2$, then for the principal branch of the logarithm, we have

$$\log z_1 = \log z_2 = \frac{2\pi i}{3},$$

and since $z_1 z_2 = e^{-2\pi i/3}$ we have

$$-\frac{2\pi i}{3} = \log(z_1 z_2) \neq \log z_1 + \log z_2.$$

Finally, for the principal branch of the logarithm the following Taylor expansion holds:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = -\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n} \quad \text{for } |z| < 1. \tag{6}$$

Indeed, the derivative of both sides equals $1/(1+z)$, so that they differ by a constant. Since both sides are equal to 0 at $z = 0$ this constant must be 0, and we have proved the desired Taylor expansion.

Having defined a logarithm on a simply connected domain, we can now define the powers z^α for any $\alpha \in \mathbb{C}$. If Ω is simply connected with $1 \in \Omega$ and $0 \notin \Omega$, we choose the branch of the logarithm with $\log 1 = 0$ as above, and define

$$z^\alpha = e^{\alpha \log z}.$$

Note that $1^\alpha = 1$, and that if $\alpha = 1/n$, then

$$(z^{1/n})^n = \prod_{k=1}^{n} e^{\frac{1}{n}\log z} = e^{\sum_{k=1}^{n}\frac{1}{n}\log z} = e^{\frac{n}{n}\log z} = e^{\log z} = z.$$

We know that every non-zero complex number w can be written as $w = e^z$. A generalization of this fact is given in the next theorem, which discusses the existence of $\log f(z)$ whenever f does not vanish.

Theorem 6.2 *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

The function $g(z)$ in the theorem can be denoted by $\log f(z)$, and determines a "branch" of that logarithm.

Proof. Fix a point z_0 in Ω, and define a function

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)}\, dw + c_0,$$

where γ is any path in Ω connecting z_0 to z, and c_0 is a complex number so that $e^{c_0} = f(z_0)$. This definition is independent of the path γ since Ω is simply connected. Arguing as in the proof of Theorem 2.1, Chapter 2, we find that g is holomorphic with

$$g'(z) = \frac{f'(z)}{f(z)},$$

and a simple calculation gives

$$\frac{d}{dz}\left(f(z)e^{-g(z)}\right) = 0\,,$$

so that $f(z)e^{-g(z)}$ is constant. Evaluating this expression at z_0 we find $f(z_0)e^{-c_0} = 1$, so that $f(z) = e^{g(z)}$ for all $z \in \Omega$, and the proof is complete.

7 Fourier series and harmonic functions

In Chapter 4 we shall describe some interesting connections between complex function theory and Fourier analysis on the real line. The motivation for this study comes in part from the simple and direct relation between Fourier series on the circle and power series expansions of holomorphic functions in the disc, which we now investigate.

Suppose that f is holomorphic in a disc $D_R(z_0)$, so that f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

that converges in that disc.

Theorem 7.1 *The coefficients of the power series expansion of f are given by*

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta}\, d\theta$$

for all $n \geq 0$ and $0 < r < R$. Moreover,

$$0 = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta}\, d\theta$$

whenever $n < 0$.