We may disregard the parts of this integral extended over $(0, \pi)$ and $\left(n\pi, \left(n+\frac{1}{2}\right)\pi\right)$, since the integrand is bounded. In view of the periodicity of $\sin u$, what remains can be written as

$$\frac{2}{\pi}\int_{\pi}^{n\pi} \frac{|\sin u|}{u}\,du = \frac{2}{\pi}\int_{0}^{\pi} (\sin u)\left(\sum_{k=1}^{n-1} \frac{1}{u+k\pi}\right)du.$$

For $0 \le u \le \pi$, the sum in brackets is contained between $\pi^{-1}\sum_{k=2}^{n}(1/k)$ and $\pi^{-1}\sum_{k=1}^{n-1}(1/k)$ and so differs from $\pi^{-1}\log n$ by an amount that is bounded in n and u. If we now note that $\int_0^{\pi}\sin u\,du = 2$, and collect estimates, we obtain $L_n = (4/\pi^2)\log n + O(1)$.

Theorem 12.37 *If f is integrable, then at each point x_0 of continuity of f,*

$$s_n\left(x_0, f\right) = o(\log n).$$

The estimate is uniform over every closed interval of continuity of f.

Proof. We will prove only the first statement, leaving the second to the reader. Suppose, as we may, that $x_0 = 0$, $f(x_0) = 0$. Because of our results about localization (see Theorem 12.34), we may assume that f vanishes outside an arbitrarily small fixed interval $(-\delta, \delta)$. Then

$$|s_n(0)| = \left|\frac{1}{\pi}\int_{-\delta}^{\delta} f(t)D_n(t)\,dt\right| \le \sup_{|t|\le\delta}|f(t)| \cdot \int_{-\pi}^{\pi}|D_n(t)|\,dt.$$

Since the sup here is small with δ and the integral is of order $\log n$, the assertion follows.

12.5 Summability of Sequences and Series

Theorem 12.35 shows that even continuous functions, when developed into Fourier series, may not be representable by those series in terms of pointwise convergence. The situation can be remedied by considering *generalized* sums of the series. This topic is vast and basic for analysis, and we will study only a few facts important for the theory of Fourier series.

Consider a fixed doubly infinite matrix of numbers (real or complex):

$$\begin{matrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0n} & \cdots \\ \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1n} & \cdots \\ & \vdots & & & \\ \alpha_{m0} & \alpha_{m1} & \cdots & \alpha_{mn} & \cdots \\ & \vdots & & & \end{matrix} \qquad (\mathscr{M})$$

Given an infinite sequence of numbers $s_0, s_1, \ldots, s_n, \ldots$, we transform it by using $(\mathscr{M})$ into a sequence $\sigma_0, \sigma_1, \ldots, \sigma_m, \ldots$ by means of the formulas

$$\sigma_m = \alpha_{m0}s_0 + \alpha_{m1}s_1 + \cdots + \alpha_{mn}s_n + \cdots \quad (m = 0, 1, 2, \ldots),$$

assuming that the series defining σ_m converges for each m. We may ask what conditions on $(\mathscr{M})$ will guarantee that whenever $\{s_n\}$ converges to a finite limit s, $\lim \sigma_m$ also exists and equals s. An answer is given by the following theorem.

Theorem 12.38 *Suppose that $(\mathscr{M})$ satisfies the following three conditions:*

(i) $\sum_n |\alpha_{mn}| \leq A$ *(for all m, with A independent of m),*
(ii) $\lim_{m\to\infty}(\sum_n \alpha_{mn}) = 1$,
(iii) $\lim_{m\to\infty} \alpha_{mn} = 0$ *for each n.*

Then for any sequence $\{s_n\}$ converging to a finite limit s, $\lim \sigma_m$ exists and equals s.

Theorem 12.38 is due to Toeplitz, and a matrix $(\mathscr{M})$ that satisfies (i)–(iii) is called a *Toeplitz matrix*.

Proof. First of all, since $\{s_n\}$ is bounded, (i) implies that σ_m exists for each m. Next, write $s_n = s + \varepsilon_n$, where $\varepsilon_n \to 0$. Then

$$\sigma_m = \sum_n \alpha_{mn}(s + \varepsilon_n) = s\sum_n \alpha_{mn} + \sum_n \alpha_{mn}\varepsilon_n.$$

We have $s\sum_n \alpha_{mn} \to s$ by (ii), and it remains only to show that the expression

$$\rho_m = \sum_n \alpha_{mn}\varepsilon_n$$

tends to 0 as $m \to \infty$. Given $\delta > 0$, split ρ_m into two sums,

$$\rho_m = \sum_{n \le n_0} \alpha_{mn} \varepsilon_n + \sum_{n > n_0} \alpha_{mn} \varepsilon_n = \rho'_m + \rho''_m,$$

say, where n_0 is so large that $|\varepsilon_n| \le \delta$ for $n > n_0$. By (i),

$$\left|\rho''_m\right| \le \sum_{n > n_0} |\alpha_{mn}| \, |\varepsilon_n| \le \sum_{n > n_0} |\alpha_{mn}| \, \delta \le A\delta.$$

On the other hand, ρ'_m consists of a fixed number of terms each of which, by (iii), tends to 0 as $m \to \infty$. Hence, $\left|\rho'_m\right| < A\delta$ for m large enough. Combining estimates, we see that $\rho_m \to 0$, which completes the proof.

It is useful to note that if $s = 0$, then condition (ii) is not required in the proof (and so in the statement of the theorem) above. It is also immediate from the proof that if $\{s_n\}$ depends on a parameter, and if $\{s_n\}$ tends uniformly to a bounded limit s, then $\{\sigma_m\}$ tends uniformly to s too.

If $\sigma_m \to s$, we shall say that the sequence $\{s_n\}$ (or the series whose partial sums are the s_n) is *summable to limit* (*sum*) *s by means of the matrix* ($\mathscr{M}$), or simply is *summable* ($\mathscr{M}$) *to s*.

The matrix ($\mathscr{M}$) is called *positive* if $\alpha_{mn} \ge 0$ for all m, n. Condition (i) is then a corollary of (ii). For positive ($\mathscr{M}$), Theorem 12.38 also holds if $s = \pm\infty$; we leave the proof to the reader.

Two methods of summability are of special significance for Fourier series.

(a) *The method of the arithmetic mean*. Given $s_0, s_1, \ldots, s_n, \ldots$, consider the *arithmetic means* $\sigma_0, \sigma_1, \ldots, \sigma_m, \ldots$ defined by

$$\sigma_m = \frac{s_0 + s_1 + \cdots + s_m}{m+1} \quad (m = 0, 1, 2, \ldots).$$

If $s_n \to s$ $(-\infty \le s \le +\infty)$, then $\sigma_m \to s$. This is clearly a special case of Theorem 12.38; the matrix is positive.

It is useful (see, e.g., the comments following the proof of Theorem 12.44) to note that if the s_n are the partial sums of a series $\sum_{k=0}^{\infty} u_k$, then

$$\sigma_m = \frac{s_0 + s_1 + \cdots + s_m}{m+1} = \frac{u_0 + (u_0 + u_1) + \cdots + (u_0 + u_1 + \cdots + u_m)}{m+1}$$
$$= \frac{1}{m+1} \sum_{k=0}^{m} (m+1-k)\, u_k.$$

Thus,

$$\sigma_m = \sum_{k=0}^{m} \left(1 - \frac{k}{m+1}\right) u_k, \quad s_m - \sigma_m = \frac{1}{m+1} \sum_{k=0}^{m} k u_k. \tag{12.39}$$

(b) *The method of Abel*. Given a series $u_0 + u_1 + \cdots + u_n + \cdots$, consider the power series

$$f(r) = \sum_{n=0}^{\infty} u_n r^n, \quad 0 \le r < 1,$$

assuming that it converges for $0 \le r < 1$. If $f(r) \to s$ as $r \to 1$, we say that $\sum u_n$ is *Abel summable* (or *A-summable*) to sum s. The method can also be applied to sequences since any sequence $\{s_n\}$ can be written as the partial sums of the series $s_0 + (s_1 - s_0) + (s_2 - s_1) + \cdots$.

Let us now see the relation of Abel summability to the general scheme. We claim that for $0 \le r < 1$, the formula

$$\sum_{n=0}^{\infty} u_n r^n = (1-r) \sum_{n=0}^{\infty} s_n r^n \quad (s_n = u_0 + \cdots + u_n) \tag{12.40}$$

is valid assuming only that one of the two series that appear is convergent. If the right side converges, it equals

$$\begin{aligned} \sum_{n=0}^{\infty} s_n r^n - \sum_{n=0}^{\infty} s_n r^{n+1} &= \sum_{n=0}^{\infty} s_n r^n - \sum_{n=1}^{\infty} s_{n-1} r^n \\ &= s_0 + \sum_{n=1}^{\infty} (s_n - s_{n-1})\, r^n = \sum_{n=0}^{\infty} u_n r^n. \end{aligned}$$

Conversely, if $\sum_{n=0}^{\infty} u_n r^n$ converges for some r, $0 < r < 1$, its Cauchy product with the absolutely convergent series $\sum_{n=0}^{\infty} r^n = (1-r)^{-1}$ converges to sum

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} u_k r^k \cdot r^{n-k} \right) = \sum_{n=0}^{\infty} (u_0 + u_1 + \cdots + u_n)\, r^n = \sum_{n=0}^{\infty} s_n r^n.$$

This proves (12.40). Now, if $\{r_m\}$ is any sequence tending to 1, $0 < r_m < 1$, then the positive numbers

$$\alpha_{mn} = (1 - r_m)\, r_m^n$$

satisfy conditions (i), (ii), (iii) of Theorem 12.38. We leave the verification to the reader.

Theorem 12.41 (Abel) *If $\sum_{n=0}^{\infty} u_n$ converges to sum s, $-\infty \le s \le +\infty$, then it is A-summable to s.*

Proof. Suppose first that s is finite. Applying (12.40), we have to show that $(1-r)\sum_{n=0}^{\infty} s_n r^n \to s$ as $r \to 1$. It is enough to prove that this relation holds for any sequence $r = r_m, m = 0, 1, \ldots$, where $0 < r_m < 1$, $r_m \to 1$. This is a corollary of Theorem 12.38 since the numbers $\alpha_{mn} = (1-r_m)r_m^n$ satisfy (i), (ii), (iii). The matrix α_{mn} is positive, and so the proof holds for $s = \pm\infty$, the only prerequisite being that the series $\sum u_n r^n$ converges for $0 \le r < 1$.

We may also consider the power series

$$f(z) = \sum_{n=0}^{\infty} u_n z^n,$$

where z is a *complex* variable lying in the unit disc: $z = re^{ix}$, $0 \le r < 1$. If $f(z)$ tends to a limit s as z tends *nontangentially* to 1, that is, as $z \to 1$ in such a way that

$$\frac{|1-z|}{1-|z|} \le C < +\infty \quad (|z| < 1),$$

then $\sum_{n=0}^{\infty} u_n$ is said to be *nontangentially Abel summable* to sum s. The last inequality means that, in approaching 1, z remains between two chords of the unit circle through $z = 1$. In fact, if $z = x + iy$ is a point that satisfies $0 < x < 1$ and $|1-z| \le C(1-|z|)$, then $|y| < C(1-x)$ since $|y| < \sqrt{y^2 + (1-x)^2} = |1-z|$ and $C(1-|z|) \le C(1-x)$. Conversely (see Exercise 23(a)), given a constant $\gamma > 0$, there are constants C and δ with $C > 0$ and $0 < \delta < 1$ such that if $z = x+iy$ with $|z| < 1, 1-\delta < x < 1$ and $|y| < \gamma(1-x)$, then $|1-z| \le C(1-|z|)$.

See (12.65) for another characterization of the notion of nontangential approach of z to 1.

Theorem 12.42 (Abel–Stolz) *If $\sum_{n=0}^{\infty} u_n$ converges to a finite sum s, then it is nontangentially Abel summable to s.*

Proof. The proof is identical to that of Abel's theorem, except that now we use the formula $\sum u_n z^n = (1-z)\sum s_n z^n$ and consider any sequence $\{z_m\}$ tending to 1 from the interior of the unit disc. The matrix α_{mn} is now $(1 - z_m)z_m^n$, conditions (ii) and (iii) of Theorem 12.38 are satisfied as before, and (i) takes the form

$$\frac{|1 - z_m|}{1 - |z_m|} \leq C.$$

Theorem 12.43 *If $\sum_{n=0}^{\infty} u_n$ is summable by the method of the arithmetic mean to sum s, then it is A-summable to s. If in addition s is finite, then $\sum_{n=0}^{\infty} u_n$ is nontangentially A-summable to s.*

Proof. Suppose that s is finite. By hypothesis,

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \to s.$$

Write $s_0 + s_1 + \cdots + s_n = t_n$. Applying formula (12.40) twice, we have

$$\sum_{n=0}^{\infty} u_n r^n = (1-r)\sum_{n=0}^{\infty} s_n r^n = (1-r)^2 \sum_{n=0}^{\infty} t_n r^n = (1-r)^2 \sum_{n=0}^{\infty} (n+1)\sigma_n r^n.$$

Again, it is enough to consider any sequence $r_m \to 1$, $0 < r_m < 1$. We then have to apply Theorem 12.38 with matrix

$$\alpha_{mn} = (1 - r_m)^2 \, (n+1) r_m^n,$$

and we easily verify that (α_{mn}) satisfies conditions (i), (ii), (iii) of Theorem 12.38. The rest of the proof of the theorem is left to the reader.

While convergence of a series implies summability A, the converse is generally false: for example, $\sum_{n=0}^{\infty}(-1)^n$ diverges but is A-summable to sum $\frac{1}{2}$ since $\sum_{n=0}^{\infty}(-r)^n = 1/(1+r) \to \frac{1}{2}$ as $r \to 1-$. If one makes additional assumptions on the terms of the series, however, the converse will hold. The following result is both elementary and useful.

Theorem 12.44 (Tauber) *If $\sum u_n$ is A-summable to sum s, $-\infty \leq s \leq +\infty$, and if $u_n = o(1/n)$ as $n \to \infty$, then $\sum u_n$ converges to sum s.*