Man bearbeite 3.2!

3

WEYL'S EQUIDISTRIBUTION THEOREM

3.1 (Weyl's criterion) Let $g: \mathbb{Z}^+ \to \mathbb{R}$ be given. Let us say that the sequence $g(n)$ is equidistributed if, for every $0 \leqslant a \leqslant b \leqslant 1$,

$$n^{-1} \operatorname{card} \{1 \leqslant r \leqslant n: a \leqslant \langle g(r) \rangle \leqslant b\} \to b - a \quad \text{as } n \to \infty.$$

Modify the proof of Theorem 3.1′ to show that the following four conditions are equivalent.

(a) g is equidistributed.

(b) $n^{-1}\sum_{r=1}^{n} f(2\pi g(r)) \to \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\,\mathrm{d}t$ as $n \to \infty$ for every continuous function $f: \mathbb{T} \to \mathbb{C}$.

(c) $n^{-1}\sum_{r=1}^{n} \exp(2\pi i m g(r)) \to 0$ as $n \to \infty$ for every integer $m \neq 0$.

(d) $n^{-1}\sum_{r=1}^{n} f(g(r)) \to \int_0^1 f(t)\,\mathrm{d}t$ as $n \to \infty$ for every continuous function $f: [0, 1] \to \mathbb{C}$.

3.2 Let $f: (0, \infty) \to \mathbb{R}$ be continuous. Question 3.1 suggests that the study of the behaviour $\sum_{r=1}^{n} \exp(2\pi i f(r))$ as $n \to \infty$ will be of great interest for number theory. This is the case, but we shall limit ourselves to a very simple example where f is slowly varying. (The arguments are due to Fejér.)

(i) Let $\phi(x) = x - [x] - 1/2$. Sketch the graph of ϕ.

(ii) Suppose f is differentiable. By integrating by parts, show that

$$\int_r^{r+1} \exp(2\pi i f(x))\,dx = 2^{-1}[\exp(2\pi i f(r+1)) + \exp(2\pi i f(r))] - 2\pi i \int_r^{r+1} \phi(x) f'(x) \exp(2\pi i f(x))\,dx.$$

(iii) By summing the previous result, show that

$$\sum_{r=1}^{n} \exp(2\pi i f(r)) = 2^{-1}[\exp(2\pi i f(n)) + \exp(2\pi i f(1))]$$
$$+ \int_1^n \exp(2\pi i f(x))\,dx$$
$$+ 2\pi i \int_1^n \phi(x) f'(x) \exp(2\pi i f(x))\,dx.$$

(iv) Conclude that, if in addition $f'(x) \to 0$ as $x \to \infty$, then

$$n^{-1} \sum_{r=1}^{n} \exp(2\pi i f(r)) - n^{-1} \int_1^n \exp(2\pi i f(x))\,dx \to 0 \quad \text{as } n \to \infty.$$

(v) Now suppose that f satisfies the previous hypotheses and, in addition, $xf'(x) \to A$ as $x \to \infty$. Show that

$$\left| \frac{1}{n} \int_1^n xf'(x) \exp(2\pi i f(x))\,dx - \frac{A}{n} \int_1^n \exp(2\pi i f(x))\,dx \right| \to 0$$

as $n \to \infty$. By integrating $\int_1^n xf'(x) \exp(2\pi i f(x))\,dx$ by parts, show using (iv) that

$$n^{-1} \sum_{r=1}^{n} \exp(2\pi i f(r)) - \frac{\exp(2\pi i f(n))}{(1 + 2\pi i A)} \to 0.$$

Conclude, using Question 3.1, that $f(n)$ is not equidistributed.

(vi) Suppose on the other hand that f satisfies the hypotheses of parts (ii) and (iv) but f' is increasing and $xf'(x) \to \infty$. Choose M such that $f'(x) > 0$ for $x > M - 1$. Explain why f has an inverse g on $(M-1, \infty)$ and why, if m is real,

$$\int_M^N \exp(2\pi i m f(x))\,dx = \int_{f(M)}^{f(N)} \exp(2\pi i m u) h(u)\,du$$

where $h(u) = 1/(f'(g(u)))$. Show that h is a decreasing continuous function and deduce, using an integral mean value theorem (see Question 52.7), that

$$\left| \int_M^N \exp(2\pi i m f(x))\,dx \right| \leqslant \sup_{f(M) \leqslant u \leqslant f(N)} |h(u)| \left| \int_{f(M)}^{f(N)} \exp(2\pi i m u)\,du \right|$$
$$\leqslant \frac{1}{f'(M)} \frac{1}{\pi |m|} \quad [m \neq 0].$$

Conclude, using Question 3.1, that $f(n)$ is equidistributed.

(vii) What can you say if $f'(x) \to 0$, $xf'(x) \to -\infty$?

(viii) For which values of α, β, γ with $\alpha, \gamma \in \mathbb{R}$, $0 \leqslant \beta < 1$ is $\alpha n^{\beta} (\log n)^{\gamma}$ equidistributed?