Übungsblatt 10

Funktionalanalysis WS 2021

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1. Let $1 < p < \infty$, and $1 < q < \infty$ be the conjugate of p, i.e., $1/p + 1/q = 1$. Let X $\left(C[0,1], \|\cdot\|_p\right)$, where $\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$ $\forall f \in C[0,1]$. Let $(b_n)_{n\in\mathbb{N}} \subseteq [0,1]$, $(c_n)_{n\in\mathbb{N}} \subseteq [0,1]$ with $b_n \leq c_n$ $\forall n \in \mathbb{N}$ and let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$. For any $n \in \mathbb{N}$ define $x_n^* : X \to \mathbb{K}$ by $x_n^*(f) = a_n \int_{b_n}^{c_n} f(x)\, dx$ $\forall f \in C[0,1]$.

i) Prove that $x_n^* \in X^*$ $\forall n \in \mathbb{N}$.

ii) Prove that the sequence $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ is pointwise bounded if and only if the sequence $(a_n(c_n - b_n))_{n\in\mathbb{N}}$ is bounded.

iii) Prove that the sequence $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ is uniformly bounded if and only if the sequence $\left(a_n(c_n - b_n)^{1/q}\right)_{n\in\mathbb{N}}$ is bounded.

iv) Prove then that we can find $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ pointwise bounded which is not uniformly bounded.

2. Let $\mathcal{R}[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is Riemann integrable}\}$ which is a real linear space with respect to the usual operations for addition and scalar multiplication. Let $\|\cdot\|_1 : \mathcal{R}[0,1] \to \mathbb{R}$ defined by $\|f\|_1 = \int_0^1 |f(x)|\, dx$ and let

$$\mathcal{N} = \{f \in \mathcal{R}[0,1] \mid \|f\|_1 = 0\}.$$

Then on the quotient space $\widetilde{\mathcal{R}}[0,1] = \mathcal{R}[0,1]/\mathcal{N}$, the map $\|\widetilde{f}\|_1 = \int_0^1 |f(x)|\, dx$ gives a structure of a normed space (for $f \in \mathcal{R}[0,1]$ we denote by $\widetilde{f} \in \widetilde{\mathcal{R}}[0,1]$ the equivalence class for f).

i) Let $g_n : [0,1] \to [0,\infty)$, $n \in \mathbb{N}$, be a sequence of continuous functions, $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, and $x_n^* : \widetilde{\mathcal{R}}[0,1] \to \mathbb{R}$ defined by $x_n^*(\widetilde{f}) = b_n \int_0^1 g_n(x) f(x)\, dx$. Prove that

a) $(x_n^*)_{n\in\mathbb{N}}$ is pointwise bounded if and only if the sequence $\left(b_n \int_0^1 g_n(x)\, dx\right)_{n\in\mathbb{N}}$ is bounded;

b) $(x_n^*)_{n\in\mathbb{N}}$ is uniformly bounded if and only if the sequence $\left(b_n \sup\limits_{x\in[0,1]} g_n(x)\right)_{n\in\mathbb{N}}$ is bounded;

ii) Let $(a_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ and $x_n^* : \widetilde{\mathcal{R}}[0,1] \to \mathbb{R}$ defined by

$$x_n^*(\widetilde{f}) = \sum_{k=1}^{n} a_k \int_0^1 x^k f(x)dx.$$

Prove that:

a) $(x_n^*)_{n\in\mathbb{N}}$ is pointwise bounded if and only if the series $\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n+1}$ converges;

b) $(x_n^*)_{n\in\mathbb{N}}$ is uniformly bounded if and only if the series $\sum\limits_{n=1}^{\infty} a_n$ converges;

iii) Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $x_n^* : \widetilde{\mathcal{R}}[0,1] \to \mathbb{R}$ defined by $x_n^*(\widetilde{f}) = a_n \int_0^1 x^n f(x)dx$. Prove that:

a) $(x_n^*)_{n\in\mathbb{N}}$ is pointwise bounded if and only if the sequence $(a_n/(n+1))_{n\in\mathbb{N}}$ is bounded;

b) $(x_n^*)_{n\in\mathbb{N}}$ is uniformly bounded if and only if the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded;

iv) Prove then that we can find $(x_n^*)_{n\in\mathbb{N}} \subseteq \left(\widetilde{\mathcal{R}}[0,1]\right)^*$ pointwise bounded, which is not uniformly bounded.