# Übungsblatt 12

Funktionalanalysis I WS 2021

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### $l_p$ (or $L_p$) is a Hilbert spaces if and only if $p = 2$

1. i) Let $1 \le p < \infty$. Prove that $\left(l_p, \|\cdot\|_p\right)$ is a Hilbert space if and only if $p = 2$.

ii) Let $1 \le p < \infty$ and $A \subseteq \mathbb{R}^n$ be a Lebesgue measurable set, $\lambda(A) > 0$ ($\lambda$ is the Lebesgue measure on $\mathbb{R}^n$). Prove that $\left(L_p(A), \|\cdot\|_p\right)$ is a Hilbert space if and only if $p = 2$.

### An inner product space which is not a Hilbert space

2. We denote by $l_0^2 = \{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{C} \mid x_n$ = . 0 onıy for a finite nu ber of $n\}$, and define $\langle\cdot,\cdot\rangle\colon l_0^2 \times l_0^2 \to \mathbb{C}$ by $\left\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \right\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$. Prove that $(l_0^2, \langle\cdot,\cdot\rangle)$ is an inner product space but not a Hilbert space.

3. i) Let $H$ be a Hilbert space and $(x_n)_{n\in\mathbb{N}} \subseteq H$ an orthonormal system. Prove that
0 *weak*.
$x_n$

ii) Let $A \subseteq [0, 2\pi]$ be a Lebesgue measurable set. Prove that

$$\lim_{n\to\infty} \int_A \sin(nt)\,dt = \lim_{n\to\infty} \int_A \cos(nt)\,dt = 0.$$

### An orthonormal set in $L_2(\mathbb{R})$

~~9~~. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{C}$ given by 4

$$f_n(x) = \pi^{-1/2} \frac{(x-i)^n}{(x+i)^{n+1}}.$$

Prove that the family $\{f_1, f_2, ...\}$ is orthonormal in $L_2(\mathbb{R})$, that is,

$$\int_{-\infty}^{\infty} f_m(x)\,\overline{f_n(x)}dx = \begin{cases} 1, & m = n, \\ 0, & m \ne n. \end{cases}$$